Triangulation of refined families

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Abstract. We prove the global triangulation conjecture for families of refined p-adic representations under a mild condition. That is, for a refined family, the associated family of (φ, Γ)-modules admits a global triangulation on a Zariski open and dense subspace of the base that contains all regular non-critical points. We also determine a large class of points which belongs to the locus of global triangulation. Furthermore, we prove that all the specializations of a refined family are trianguline. In the case of the Coleman–Mazur eigencurve, our results provide the key ingredient for showing its properness in a subsequent work [15].

Contents

1. Introduction

In the seminal work [25], Kisin proved the Fontaine–Mazur conjecture for Galois representations attached to finite slope, overconvergent cuspidal eigenforms. The most significant part of his proof is showing that the dual of these representations satisfy the property that their restrictions on a decomposition group of p have nonzero crystalline periods on which the crystalline Frobenius acts via multiplication with the U_p-eigenvalue. Furthermore, he conjectured that this property should characterize the Galois representations coming from finite slope overconvergent p-adic modular forms. This beautiful result inspired many important subsequent developments. In p-adic Hodge theory, Colmez introduced the notion of *trianguline representations* reformulating this property in the framework of (φ, Γ)-modules over the Robba ring [13]; the notion of trianguline representations plays a key role in his construction of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. In the direction of Bloch–Kato conjecture, Bellaiche–Chenevier [1] and Skinner–Urban [33] applied some (different) variants of Kisin's result to construct elements of Selmer groups by deforming certain p-adic representations on eigenvarieties. More recently, Emerton [16] established the local-global compatibility of p-adic Langlands for $\mathrm{GL}_2/\mathbb{Q}$. As an application, he confirmed the conjecture of Kisin. Nowadays, it is widely assumed that the condition of being trianguline at the places above p characterizes the Galois representations coming from finite slope overconvergent p-adic automorphic forms. In addition, it is conjectured[1] that for a family of p-adic representations arising on eigenvarieties, the associated family of (φ, Γ)-modules admits a *global triangulation* on a Zariski open and dense subspace of the base that contains all non-critical points.

The main objects of this paper are *families of refined p-adic representations*. This notion was first introduced by Bellaiche–Chenevier [1] for p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ to encode the properties of families of Galois representations carried

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[1]See for instance [1, §4].

by eigenvarieties. In this paper, we first generalize this notion to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$-representations where K is a finite extension over $\mathbb{Q}_p$. For technical reasons, we will assume throughout that our refined families are arithmetic families of p-adic representations, not just pseudocharacters as in Bellaiche–Chenevier's original definition. The main goal of this paper is to prove the global triangulation conjecture for such families under a mild condition. Namely, we will prove that a family of refined p-adic representations admits a global triangulation on a Zariski open and dense subspace of the base that contains all regular non-critical points. We also determine a large class of points which belongs to the locus of global triangulation. Furthermore, we will show that the specializations of refined families are all trianguline. Finally, as an application, we explicitly determine the local behavior of the family of p-adic representations carried by the Coleman–Mazur eigencurve.

Our approach is largely inspired by Kisin's method of interpolating crystalline periods [25]. The first major step is to show that for a weakly refined family, the de Rham periods always coincide with the crystalline periods on which the Frobenius acts via multiplication with the prescribed eigenvalue. Furthermore, both of them form coherent sheaves on the base. To this end, we significantly refine Kisin's construction of finite slope subspaces by removing the "Y-small" assumption. The value of this refinement is that it allows us to interpolate periods over all affinoid subdomains of the base, not only, as in the case of Kisin's original construction, over Y-small affinoid subdomains.

The second major step is to show that for a given refined family, the crystalline periods of its exterior powers give rise to the desired global triangulation. Firstly, it is not difficult to see that the set of points by which the crystalline periods of exterior powers of the family give rise to a triangulation of the specialization, which is named as the *triangulation locus* of the family, is an analytic subspace of a Zariski open subspace of the base. Therefore, to prove the global triangulation conjecture, it reduces to show that the triangulation locus contains all regular non-critical points. We achieve this by combining the results of the first step and a result of Bellaiche–Chenevier on descent [1].

Indeed, by the recent works of Bellovin [6] and Kedlaya–Pottharst–Xiao [24], one knows that for an arithmetic family of p-adic representations over a rigid analytic space, the de Rham periods and crystalline periods on which the Frobenius acts via multiplication with an invertible function always form coherent sheaves on the base respectively. On the other hand, as mentioned above, we prove in this paper that for a weakly refined family, the de Rham periods always coincide with the crystalline periods on which the Frobenius acts via multiplication with the prescribed eigenvalue. This fact is a special feature for the eigen-families of p-adic representations, and turns out to be important for applications to eigenvarieties. For example, it follows that for the dual of the family of Galois representations carried by the Coleman–Mazur eigencurve, the de Rham periods coincide with the crystalline periods on which the Frobenius acts via multiplication with the U_p-eigenvalue.

This fact plays a key role in our subsequent work with Hansheng Diao proving the properness of the Coleman–Mazur eigencurve [15].

Last but not least, we should point out that after the work of this paper was finished, some of our results were also obtained by other authors. In [18], Hellmann proves that the families of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representations carried by eigenvarieties for definite unitary groups over imaginary quadratic fields admit global triangulations as in our case. His strategy is to construct a map from the eigenvariety to the moduli space of (rigidified) trianguline (φ, Γ)-modules. In [24], Kedlaya–Pottharst–Xiao establish the finiteness of cohomology for arithmetic families of (φ, Γ)-modules. As an application, they show that an arithmetic family of p-adic representations that is densely pointwise refined in the sense of Mazur admits a global triangulation over a large subspace of the base. More recently, in his thesis [7], John Bergdall applies the techniques of [24] to refined families and gives a new proof of our result that the triangulation locus contains all regular non-critical points in the case of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representations.

In the following we will explain the main results of the paper and the idea of proofs in more detail.

1.1. Finite slope subspaces. We fix a finite extension K of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$, and fix a uniformizer π_K of K. Let $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. Let K_0 be the maximal unramified extension of $\mathbb{Q}_p$ contained in K, and let $f = [K_0 : \mathbb{Q}_p]$. Let E be the Galois closure of K in $\overline{\mathbb{Q}}_p$, and let H_K be the set of $\mathbb{Q}_p$-embeddings of K into $\overline{\mathbb{Q}}_p$ (hence into E). For a $K \otimes_{\mathbb{Q}_p} E$-module M, set

$$M_\tau = M \otimes_{K \otimes_{\mathbb{Q}_p} E} (K \otimes_{K,\tau} E)$$

for any $\tau \in \mathrm{H}_K$. We may identify M with $\oplus_{\tau \in \mathrm{H}_K} M_\tau$. For any $m \in M$, let m_τ denote the projection of m onto M_τ.

We now assume that X is a rigid analytic space over E, and let V_X be a family of p-adic representations of G_K of dimension d over X. Suppose the Sen polynomial[2] for V_X is $TQ(T)$ for some $Q(T) \in (K \otimes_{\mathbb{Q}_p} \mathcal{O}(X))[T]$. For $\tau \in \mathrm{H}_K$, following the notation introduced above, $Q(T)_\tau$ denotes the projection of $Q(T)$ onto the τ-isotypic component of $(K \otimes_{\mathbb{Q}_p} \mathcal{O}(X))[T] = K \otimes_{\mathbb{Q}_p} \mathcal{O}(X)[T]$. Let $\alpha \in \mathcal{O}(X)^\times$ be an invertible rigid analytic function on X. We define finite slope subspaces of X with respect to the pair (α, V_X) as follows.

Definition 1.1. For such a triple (X, α, V_X), we call an analytic subspace[3] $X_{fs} \subset X$ a *finite slope subspace of X with respect to the pair (α, V_X)* if it satisfies the following conditions.

(1) For every integer $j \leq 0$ and $\tau \in \mathrm{H}_K$, the subspace $(X_{fs})_{Q(j)_\tau}$, which denotes the complement of the vanishing locus of $Q(j)_\tau$ on X_{fs}, is Zariski open and dense in X_{fs}.

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[2]See Definition 2.24 for the definition of Sen polynomial.
[3]In this paper, the terminology "analytic subspaces" refers to Zariski closed rigid analytic subspaces.

(2) For any morphism $g : M(R) \to X$, where R is an E-affinoid algebra, if g factors through $X_{Q(j)_\tau}$ for every integer $j \leq 0$ and $\tau \in H_K$, then it factors through X_{fs} if and only if the natural map

$$\iota_{n,K} : K \otimes_{K_0} (\mathrm{D}^\dagger_{\mathrm{rig}}(g^*(V_X))^{\varphi^f = g^*(\alpha), \Gamma = 1} \to \mathrm{D}^{+,fn}_{\mathrm{dif}}(g^*(V_X))^\Gamma \qquad (1.1)$$

is an isomorphism for all sufficiently large n (indeed for all $n \geq n(V_R))$[4].

The above definition is a (φ, Γ)-module theoretical interpretation of Kisin's original definition of finite slope subspaces except that we relax the assumption on g. That is, we do not require that g is α-small in the sense of [25, (5.2)]. It is not difficult to see that our finite slope subspace X_{fs} (assuming its existence and uniqueness) coincides with the one introduced by Nakamura [31] which generalizes of Kisin's finite slope subspaces to finite extensions of $\mathbb{Q}_p$ (see Remark 4.14).

The idea for introducing finite slope subspaces is to cut out the maximal analytic subspace X_{fs} of X such that $Q(j)_\tau$ is not identically 0 on any component of X_{fs} for any $j \leq 0$ and $\tau \in H_K$, and for any affinoid subdomain $M(S)$ of X, the natural maps

$$K \otimes_{K_0} \mathrm{D}^\dagger_{\mathrm{rig}}(V_X|_{M(S) \cap X_{fs}})^{\varphi^f = \alpha, \Gamma = 1} \to (\mathrm{D}^{+,fn}_{\mathrm{dif}}(V_X|_{M(S) \cap X_{fs}})/(t^k))^\Gamma \qquad (1.2)$$

are isomorphisms for all sufficiently large k. As it was already pointed out by Kisin [25, (5.5)(5)], the "Y-small" assumption in [25, Proposition 5.4] is due to some technical obstacle to solve a certain Frobenius equation over the relative crystalline period ring. We get over this difficulty by using the relative extended Robba ring which is much bigger than the relative crystalline period ring and sufficient to solve this equation.

Theorem 1.2 (Theorem 4.10). *The rigid analytic space X has a unique finite slope subspace with respect to the pair (α, V_X).*

More importantly, we will prove that if k is bigger than the valuation of α in S, then (1.2) is an isomorphism. This result is crucial for later applications to refined families.

Theorem 1.3 (Theorem 4.12). *Let $M(S)$ be an affinoid subdomain of X_{fs}. Then for any $n \geq n(V_S)$ and $k > \log_{|\pi_K^{-1}|} |\alpha^{-1}|_{\mathrm{sp}}$ where the spectral norm is taken in S, the natural map*

$$K \otimes_{K_0} \mathrm{D}^\dagger_{\mathrm{rig}}(V_X|_{M(S)})^{\varphi^f = \alpha, \Gamma = 1} \to (\mathrm{D}^{+,fn}_{\mathrm{dif}}(V_X|_{M(S)})/(t^k))^\Gamma$$

is an isomorphism. As a consequence, the presheaf $M(R) \mapsto \mathrm{D}^\dagger_{\mathrm{rig}}(V_X|_{M(R)})^{\varphi^f = \alpha, \Gamma = 1}$, where $M(R)$ runs through all affinoid subdomains of $M(S)$, is indeed a coherent sheaf on $M(S)$.

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[4] See §2.3 for the definition of the functor $\mathrm{D}^{+,n}_{\mathrm{dif}}$.

We denote by $\mathscr{D}^\dagger_{\mathrm{rig}}(V_S)^{\varphi^f=\alpha,\Gamma=1}$ this coherent sheaf, and by $\mathscr{D}^\dagger_{\mathrm{rig}}(V_{X_{fs}})^{\varphi^f=\alpha,\Gamma=1}$ the coherent sheaf on X_{fs} obtained by gluing the sheaves $\mathscr{D}^\dagger_{\mathrm{rig}}(V_S)^{\varphi^f=\alpha,\Gamma=1}$ for all affinoid subdomains $M(S)$ of X_{fs}.

1.2. Rank 1 (φ,Γ)-modules and trianguline representations.

For a $K_0 \otimes_{\mathbb{Q}_p} K_0$-module M, set

$$M_\sigma = M \otimes_{K_0 \otimes_{\mathbb{Q}_p} K_0} (K_0 \otimes_{K_0,\sigma} K_0)$$

for any $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$. We may identify M with $\bigoplus_{\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)} M_\sigma$. Then for any $m \in M$, let m_σ denote the M_σ-component of it. Now let S be an affinoid algebra over K_0, set $S_{K_0} = S \otimes_{\mathbb{Q}_p} K_0$. Let $\phi \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$ be the arithmetic Frobenius. Using the canonical isomorphism

$$S_{K_0} \cong \prod_{0 \le i \le f-1} S_{\phi^i},$$

for any $a \in S$ we equip S_{K_0} with a $1 \otimes \phi$-semilinear action φ by setting

$$\varphi(x_1, x_2, \ldots, x_{f-1}) = (a x_{f-1}, x_1, \ldots, x_{f-2}).$$

Let D_a denote this φ-module. The φ-action on D_a satisfies $\varphi^f = a \otimes 1$.

Let K'_0 be the maximal unramified extension of $\mathbb{Q}_p$ contained in $K(\mu_{p^\infty})$. Recall that one may identify $\mathbf{B}^\dagger_{\mathrm{rig},K}$ with the Robba ring $\mathcal{R}_{K'_0}$ over K'_0 (see [2, §2.6]). Equip $\mathcal{R}_{K'_0}$ with the induced actions of φ and Γ. Let $\widehat{\mathscr{T}}(S)$ be the set of continuous characters $\delta : K^\times \to S^\times$. For any $\delta \in \widehat{\mathscr{T}}(S)$, we attach to it a rank 1 (φ,Γ)-module $\mathcal{R}_S(\delta)$ over $\mathcal{R}_{K'_0}\widehat{\otimes}_{\mathbb{Q}_p}S$ as follows. If $\delta|_{\mathcal{O}_K^\times}$ is trivial, we set $\mathcal{R}_S(\delta) = D_{\delta(\pi_K)} \otimes_{S_{K_0}} (\mathcal{R}_{K'_0}\widehat{\otimes}_{\mathbb{Q}_p}S)$; here $D_{\delta(\pi_K)}$ is equipped with the trivial Γ-action. For general δ, we may write $\delta = \delta'\delta''$ so that $\delta'(\pi_K) = 1$ and $\delta''|_{\mathcal{O}_K^\times}$ is the trivial character. By local class field theory, δ' can be viewed as an $S^\times$-valued continuous character of the Weil group of K; it extends to a character of G_K by continuity. We set $\mathcal{R}_S(\delta) = \mathrm{D}^\dagger_{\mathrm{rig}}(\delta') \otimes_{\mathcal{R}_{K'_0}\widehat{\otimes}_{\mathbb{Q}_p}S} \mathcal{R}_S(\delta'')$. For any (φ,Γ)-module D_S over $\mathcal{R}_{K'_0}\widehat{\otimes}_{\mathbb{Q}_p}S$, set $D_S(\delta) = D_S \otimes_{\mathcal{R}_{K'_0}\widehat{\otimes}_{\mathbb{Q}_p}S} \mathcal{R}_S(\delta)$.

Definition 1.4. For $\delta \in \widehat{\mathscr{T}}(S)$, a rank 1 (φ,Γ)-module over $\mathcal{R}_{K'_0}\widehat{\otimes}_{\mathbb{Q}_p}S$ is called of *type* δ if it is isomorphic to $M \otimes_{S_{K_0}} \mathcal{R}_S(\delta)$ for some locally free rank 1 S_{K_0}-module M equipped with trivial φ- and Γ-actions. We call a (φ,Γ)-module D_S over $\mathcal{R}_{K'_0}\widehat{\otimes}_{\mathbb{Q}_p}S$ *triangulable* if it admits a filtration

$$0 = \mathrm{Fil}_0(D_S) \subset \mathrm{Fil}_1(D_S) \subset \cdots \subset \mathrm{Fil}_{d-1}(D_S) \subset \mathrm{Fil}_d(D_S) = D_S$$

by (φ,Γ)-submodules over $\mathcal{R}_{K'_0}\widehat{\otimes}S$ such that each successive quotient

$$\mathrm{Fil}_i(D_S)/\mathrm{Fil}_{i-1}(D_S)$$

is of type δ_i for some $\delta_i \in \widehat{\mathscr{T}}(S)$; any such a filtration $(\mathrm{Fil}_i(D_S))_{1 \le i \le d}$ is called a *triangulation* of D_S, and $(\delta_i)_{1 \le i \le d}$ are called the *parameters* of this

triangulation. We call a locally free S-linear representation V_S of G_K *trianguline* if the corresponding (φ, Γ)-module $D^\dagger_{\mathrm{rig}}(V_S)$ is triangulable.

1.3. Refined and weakly refined families. From now on, let X be reduced. For any $x \in X$, let V_x denote the specialization of V_X at x. The following definitions generalize Bellaiche–Chenevier's notions of refined and weakly refined families [1, §4.2.3] to p-adic representations of G_K[5]. As previously mentioned, we will define our refined and weakly refined families to be arithmetic families of p-adic representations, not just pseudocharacters as in Bellaiche–Chenevier's original definitions.

Definition 1.5. A *family of weakly refined p-adic representations* of G_K of dimension d over X is a family of p-adic representations V_X of G_K of dimension d over X together with the following data

  (1) d analytic functions $\kappa_1, \ldots, \kappa_d \in K \otimes_{\mathbb{Q}_p} \mathcal{O}(X)$,

  (2) an analytic function $F \in \mathcal{O}(X)$,

  (3) a Zariski-dense subset Z of X,

subject to the following requirements.

  (a) For every $x \in X$, the generalized Hodge–Tate weights[6] of V_x are, with multiplicity, $\kappa_1(x), \ldots, \kappa_d(x)$.

  (b) If $z \in Z$, V_z is crystalline.

  (c) If $z \in Z$, $\kappa_1(z)_\tau$ is the biggest Hodge–Tate weight of $D_{\mathrm{dR}}(V_z)_\tau$ for every $\tau \in \mathrm{H}_K$.

  (d) For each $z \in Z$, $D_{\mathrm{crys}}(V_z)$ has a φ-submodule over $K_0 \otimes_{\mathbb{Q}_p} k(x)$ which is isomorphic to $D_{F(z) \prod_{\tau \in \mathrm{H}_K} \tau(\pi_K)^{-\kappa_1(z)_\tau}}$.

  (e) For any non-negative integer C, let Z_C be the set

  $$\{z \in Z, \kappa_1(z)_\tau - \kappa_n(z)_\tau > C, \forall n \in \{2, \ldots, d\}, \tau \in \mathrm{H}_K\}.$$

  Then Z_C accumulates at any $z \in Z$ for all C.

  (f) There exists a continuous character $\chi : \mathcal{O}_K^\times \to \mathcal{O}(X)^\times$ whose derivative at 1 is $-\kappa_1$ and whose evaluation at any $z \in Z$ is the character

  $$x \mapsto \prod_{\tau \in \mathrm{H}_K} \tau(x)^{-\kappa_1(z)_\tau}.$$

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[5]Note that the Hodge–Tate weight of p-adic cyclotomic character is normalized to be -1 in [1]; this is opposite to our normalization.

[6]We set the generalized Hodge–Tate weights to be the roots of the Sen polynomial.

We may also view χ as a continuous character from $K^\times$ to $\mathcal{O}(X)^\times$ by setting $\chi(\pi_K) = 1$. By twisting V_X with χ^{-1}, we may suppose $\kappa_1 = 0$. In this case, we have the following result.

Theorem 1.6 (Theorem 5.3). *The finite slope subspace of X with respect to the pair (F, V_X) is X itself.*

Definition 1.7. A *family of refined p-adic representations* of G_K of dimension d over X is a family of p-adic representations V_X of G_K of dimension d over X together with the following data

(1) d analytic functions $\kappa_1, \ldots, \kappa_d \in K \otimes_{\mathbb{Q}_p} \mathcal{O}(X)$,

(2) d analytic functions $F_1, \ldots, F_d \in \mathcal{O}(X)$,

(3) a Zariski-dense subset Z of X,

subject to the following requirements.

(a) For every $x \in X$, the generalized Hodge–Tate weights of V_x are, with multiplicity, $\kappa_1(x), \ldots, \kappa_d(x)$.

(b) If $z \in Z$, V_z is crystalline.

(c) If $z \in Z$, then $\kappa_1(z)_\tau > \kappa_2(z)_\tau > \cdots > \kappa_d(z)_\tau$ for any $\tau \in H_K$.

(d) For each $z \in Z$, there exists a refinement of V_z such that the associated ordering of the φ^f-eigenvalues are

$$\left(\prod_{\tau \in H_K} \tau(\pi_K)^{-\kappa_1(z)_\tau} F_1(z), \ldots, \prod_{\tau \in H_K} \tau(\pi_K)^{-\kappa_d(z)_\tau} F_d(z) \right).$$

(e) For any non-negative integer C, let Z_C be the set

$$\{ z \in Z, |\kappa_I(z)_\tau - \kappa_J(z)_\tau| > C, \forall I, J \subseteq \{1, \ldots, d\},$$
$$|I| = |J| > 0, I \neq J, \tau \in H_K \},$$

where $\kappa_I = \sum_{i \in I} \kappa_i$. Then Z_C accumulates at any $z \in Z$ for all C.

(f) For each $1 \le i \le d$, there exists a continuous character $\chi_i : \mathcal{O}_K^\times \to \mathcal{O}(X)^\times$ whose derivative at 1 is $-\kappa_i$ and whose evaluation at any $z \in Z$ is the character

$$x \mapsto \prod_{\tau \in H_K} \tau(x)^{-\kappa_i(z)_\tau}.$$

For each $1 \le i \le d$, let $\alpha_i = \prod_{j=1}^i F_j$ and $\eta_i = \prod_{j=1}^i \chi_i$. Let δ_i be the continuous character $\delta_i : K^\times \to \mathcal{O}(X)^\times$ defined by $\delta_i(\pi_K) = \alpha_i$ and $\delta_i|_{\mathcal{O}_K^\times} = \eta_i$. It is straightforward to see that $\wedge^i V_X$ is weakly refined with $F = \alpha_i$ and $\chi = \eta_i$. Thus by Theorem 1.6 and Theorem 1.2, for each $1 \le i \le d$, we get a coherent sheaf

of crystalline periods $\mathscr{D}^\dagger_{\mathrm{rig}}((\wedge^i V_X)(\eta_i^{-1}))^{\varphi^f=\alpha_i,\Gamma=1}$ on X. The main result of this paper is the following theorem (see Theorem 5.42 for a more precise version). Here for any $\mathcal{O}(X)^\times$-valued character Ψ and $x \in X$, we denote by $\Psi(x)$ the evaluation of Ψ at x.

Theorem 1.8. *The families of* (φ, Γ)-*modules*

$$\mathscr{D}^\dagger_{\mathrm{rig}}((\wedge^i V_X)(\eta_i^{-1}))^{\varphi^f=\alpha_i,\Gamma=1} \otimes_{K_0 \otimes \mathcal{O}_X} \mathscr{D}^\dagger_{\mathrm{rig}}(\eta_i)$$

for all $1 \leq i \leq d$ *give rise to a global triangulation of* V_X *on a Zariski open and dense subspace of* X *with parameters* $(\delta_i/\delta_{i-1})_{1 \leq i \leq d}$. *Furthermore, the locus of global triangulation contains all* $x \in X$ *such that* $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$ *admits a triangulation with parameters* $((\delta_i/\delta_{i-1})(x))_{1 \leq i \leq d}$ *and satisfies*

$$\dim_{\mathbb{Q}_p} \mathrm{D}^\dagger_{\mathrm{rig}}((\wedge^i V_x^{\mathrm{ss}})(\eta_i^{-1}(x)))_\sigma^{\varphi^f=\alpha_i(x),\Gamma=1} = 1$$

for every $1 \leq i \leq d-1$ *and* $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$. *In particular, the locus of global triangulation contains all regular non-critical points.*

1.4. Triangulation loci of refined families.

Definition 1.9. Let V_X be a weakly refined family. For $x \in X$, we say x is *saturated* for the family V_X if the following two conditions hold.

(1) For any $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, the coherent $\mathcal{O}_X$-module $\mathscr{D}^\dagger_{\mathrm{rig}}(V_X(\chi^{-1}))_\sigma^{\varphi^f=F,\Gamma=1}$ is locally free of rank 1 around x.

(2) The image of $\mathscr{D}^\dagger_{\mathrm{rig}}(V_X(\chi^{-1}))^{\varphi^f=F,\Gamma=1}$ in $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x(\chi^{-1}(x)))$ generates a rank 1 saturated (φ, Γ)-submodule.

We denote by X_s the set of saturated points, and call it the *saturated locus* of V_X.

Using Theorem 1.3, it is not difficult to show that X_s is a Zariski open and dense subspace of X. For a refined family V_X, the saturated locus X_s is defined to the intersections of the saturated loci of the weakly refined families $\wedge^i V_X$ for all $1 \leq i \leq d$. It follows that X_s is a Zariski open and dense subspace of X.

Definition 1.10. Let V_X be a refined family. The *triangulation locus* of V_X is defined to be the set of $x \in X_s$ such that the (φ, Γ)-modules

$$\mathscr{D}^\dagger_{\mathrm{rig}}((\wedge^i V_X)(\eta_i^{-1}))^{\varphi^f=\alpha_i,\Gamma=1} \otimes_{K_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_X} \mathrm{D}^\dagger_{\mathrm{rig}}(\eta_i(x))$$

for all $1 \leq i \leq d$ give rise to a triangulation of $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$. That is, there exists a triangulation $(\mathrm{Fil}_i(\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)))_{1 \leq i \leq d}$ of $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$ such that

$$\mathscr{D}^\dagger_{\mathrm{rig}}((\wedge^i V_X)(\eta_i^{-1}))^{\varphi^f=\alpha_i,\Gamma=1} \otimes_{K_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_X} \mathrm{D}^\dagger_{\mathrm{rig}}(\eta_i(x)) = \wedge^i(\mathrm{Fil}_i(\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)))$$

for all $1 \leq i \leq d$.

It is obvious that the locus of global triangulation is contained in the triangulation locus. In fact, it turns out that they actually coincide. That is, a refined family admits a global triangulation on the triangulation locus.

Proposition 1.11 (Proposition 5.40). *The triangulation locus forms a reduced Zariski closed subspace of X_s. Furthermore, for any affinoid subdomain $M(S)$ of the triangulation locus, the sequence of (φ, Γ)-modules*

$$(D^{\dagger}_{\mathrm{rig}}((\wedge^i V_X|_{M(S)})(\eta_i^{-1}))^{\varphi^f = \alpha_i, \Gamma = 1} \otimes_{K_0 \otimes_{\mathbb{Q}_p} S} D^{\dagger}_{\mathrm{rig}}(\eta_i))_{1 \le i \le d}$$

gives rise to a triangulation of $D^{\dagger}_{\mathrm{rig}}(V_X|_{M(S)})$ with parameters $(\delta_i/\delta_{i-1})_{1 \le i \le d}$.

To cut out the triangulation locus, we view the associated family of (φ, Γ)-modules as a family of vector bundles over the relative half-open annulus $X \times \{0 < v_p(T) \le r\}$ for some $r > 0$. For a general base X, it is difficult to deal with vector bundles over such a relative annulus. We get over this difficulty by restricting the family of vector bundles on a closed annulus $v_p(T) \in [r/p^f, r]$, where r is sufficiently small, over X. We then cut out the triangulation locus and construct the global triangulation over this closed annulus. Finally, we use the Frobenius action to extend the global triangulation over the closed annulus to a global triangulation of the original family of (φ, Γ)-modules.

Therefore, to prove the global triangulation conjecture, it finally reduces to show that all regular non-critical points belong to the triangulation locus. As mentioned before, we prove this using a result of Bellaiche–Chenevier on descent [1, §3.2]. On the other hand, although the global triangulation can not be extended to the whole base (because of the existence of critical points as pointed out by Bellaiche–Chenevier [1, Remark 2.5.9]), it turns out that the specializations of refined families are all trianguline.

Theorem 1.12 (Theorem 5.45). *For any $x \in X$, V_x is trianguline.*

1.5. Application to the eigencurve. Let $\mathcal{C}$ be the eigencurve associated with an absolutely irreducible 2-dimensional residual representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is p-modular in the sense of [11]. Let $\alpha \in \mathcal{O}(\mathcal{C})^{\times}$ be the function of U_p-eigenvalues, and let $\kappa : \mathcal{C} \to \mathcal{W}$ be the map to the weight space. We normalize κ in such a way that if x is a classical eigenform of weight k, then $\kappa(x) = k - 1$. Let

$$\epsilon : (\mathbb{Z}/N\mathbb{Z})^{\times} \times \mathbb{Z}_p^{\times} \to \mathcal{O}(\mathcal{C})^{\times}$$

be the nebentypus-weight character (cf. [11, §3.1]). That is, the diamond operators act on overconvergent eigenforms parametrized by $\mathcal{C}$ through ϵ.

Following [25], let $V_{\mathcal{C}}$ be the *dual* of the family of p-adic representations of $G_{\mathbb{Q}}$ on $\mathcal{C}$ interpolating the Galois representations attached to classical eigenforms. That is, for any $x \in \mathcal{C}$ and prime l not dividing pN, the characteristic polynomial of the

geometric Frobenius at l on V_x is

$$X^2 - a_l(x)X + \epsilon(x) = 0,$$

where a_l denotes the l-th coefficient of the q-expansion. Let Z be the set of classical points $z \in \mathcal{C}$ such that V_z is crystalline with distinct crystalline Frobenius eigenvalues. Coleman's classicality theorem then ensures that $V_{\mathcal{C}}$ is a weakly refined family together with $\kappa_1 = 0, \kappa_2 = -\kappa$, $F = \alpha$ and Z. The following theorem completely determines the local behavior of $V_{\mathcal{C}}$.

Theorem 1.13 (Theorem 5.48). *For any $x \in \mathcal{C}$, the coherent sheaf $\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_{\mathcal{C}})^{\varphi=F,\Gamma=1}$ is locally free of rank 1 around x unless $\kappa(x) = 0$ and $\dim D_{\mathrm{crys}}(V_x^{\mathrm{ss}})^{\varphi=F(x)} = 2$. In particular, V_x^{ss} is crystalline in this case. If x is not of this form, it is not saturated if and only if it satisfies one of the following two disjoint conditions.*

(1) *The weight $\kappa(x)$ is a positive integer and $v_p(F(x)) > \kappa(x)$. As a consequence, V_x belongs to $\mathscr{S}_*^{\mathrm{ng}} \cap \mathscr{S}_*^{\mathrm{HT}}$ in the sense of [14]; hence V_x is irreducible, Hodge–Tate and non-de Rham. Furthermore, the image of $t^{-\kappa(x)}(\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_{\mathcal{C}}))^{\varphi=F,\Gamma=1}$ generates a rank 1 saturated (φ, Γ)-submodule in $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_x)$.*

(2) *The weight $\kappa(x)$ is a positive integer and $v_p(F(x)) = \kappa(x)$, and V_x has a rank 1 subrepresentation V_x' which is crystalline with Hodge–Tate weight $-\kappa(x)$. Furthermore, in this case, the image of $\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_{\mathcal{C}})^{\varphi=F,\Gamma=1}$ in $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_x)$ is $k(x) \cdot t^{\kappa(x)} e'$ where e' is a canonical basis of $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_x')$.*

In case (2), if $x \in Z$, then it is critical. Hence it is decomposable. Suppose $V_x = V_1 \oplus V_2$ where V_1 has Hodge–Tate weight 0 and V_2 has Hodge–Tate weight $-\kappa(x)$. Then the image of $\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_{\mathcal{C}})^{\varphi=F,\Gamma=1}$ in $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_x)$ is $k(x) \cdot t^{\kappa(x)} e_2$ where e_2 is a canonical basis of $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_2)$.

Notation and conventions. Let v_p denote the p-adic valuation on $\mathbb{C}_p$ normalized as $v_p(p) = 1$. Let $|\cdot|$ be the corresponding norm defined by $|x| = p^{-v_p(x)}$. Fix a finite extension K of $\mathbb{Q}_p$ in $\mathbb{C}_p$. Let $\mathcal{O}_K$ be the ring of integers of K, and let π_K be a fixed uniformizer. Let v_K denote the p-adic valuation on $\mathbb{C}_p$ normalized as $v_K(\pi_K) = 1$. For any valuation v (norm $|\cdot|$) and a matrix $A = (A_{ij})$, we use $v(A)$ (resp. $|A|$) to denote the minimal valuation (resp. maximal norm) among the entries.

We may view any continuous character of $\mathcal{O}_K^\times$ as a continuous character of $K^\times$ by pulling back via the projection $K^\times \to \mathcal{O}_K^\times$ determined by π_K. We may further view it as a continuous character of W_K, which denotes the Weil group of K, via the local reciprocity isomorphism $W_K^{\mathrm{ab}} \cong K^\times$ where a geometric Frobenius element maps to π_K.

We choose a compatible sequence of primitive p-powers roots of unity $(\varepsilon_n)_{n\geq0}$, i.e. each $\varepsilon_n \in \overline{\mathbb{Q}}_p$ is a primitive p^n-th root of 1, and they satisfy $\varepsilon_{n+1}^p = \varepsilon_n$ for all $n \geq 0$. Fix $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots)$ to be Fontaine's p-adic $\exp(2\pi i)$. For a finite extension L of $\mathbb{Q}_p$ in $\mathbb{C}_p$, let $L_n = L(\varepsilon_n)$ for $n \geq 1$, and let $L_\infty = \cup_{n\in\mathbb{N}}L_n$. Let L'_0 be the maximal unramified extension of $\mathbb{Q}_p$ in L_∞. Let $H_L = \mathrm{Gal}(\overline{\mathbb{Q}}_p/L_\infty)$, and let $\Gamma_L = \mathrm{Gal}(L_\infty/L)$. Denote Γ_K by Γ for simplicity.

We normalize the Hodge–Tate weight in a way that the p-adic cyclotomic character has Hodge–Tate weight 1.

Let K_0 be the maximal unramified extension of K in $\mathbb{C}_p$, and let $f = [K_0 : \mathbb{Q}_p]$. For any $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, let H_σ be the set of $\tau \in \mathrm{H}_K$ such that its restriction on K_0 is σ.

For $r > 0$, put $\rho(r) = \frac{p-1}{pr}$. For $n \geq 0$, let $r_n = p^{n-1}(p-1)$. For $s > 0$, let $n(s)$ be the maximal integer n such that $r_n \leq s$.

For an affinoid algebra S, we denote by $\mathcal{O}_S$ the unit ball of S. For a topological group G and a rigid analytic space X over $\mathbb{Q}_p$, by a *family of p-adic representations of G of dimension d on X* we mean a locally free coherent $\mathcal{O}_X$-module V_X of rank d equipped with a continuous $\mathcal{O}_X$-linear G-action. When $X = M(S)$ is an affinoid space over $\mathbb{Q}_p$, we also call a family of p-adic representations of G on X an *S-linear G-representation*. If $M(R) \subset M(S)$ is an affinoid subdomain and V_S is a family of representation on $M(S)$, write V_R for the base change of V_S from S to R. Finally, for every $x \in M(S)$, we write V_x to denote the specialization $V_S \otimes_S k(x)$.

2. Preliminaries

2.1. The (φ, Γ)-module functor. Let S be an affinoid algebra over $\mathbb{Q}_p$, and let V_S be a finite locally free S-linear representation of G_K. The (φ, Γ)-module functors $\mathrm{D}_K^\dagger(V_S)$ and $\mathrm{D}_{\mathrm{rig},K}^\dagger(V_S)$ are constructed in [4] and [22]. However, both of these works do not really verify that $\mathrm{D}_K^\dagger(V_S)$ and $\mathrm{D}_{\mathrm{rig},K}^\dagger(V_S)$ are φ-modules in the sense that they are isomorphic to their φ-pullbacks respectively. This small gap will be filled in this subsection. We follow the notations of [4] and [22], and refer the reader to them for more details. Recall that Berger–Colmez show that the ring $\widetilde{\mathbf{A}}^{(0,1]}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_S$ together with the cyclotomic character $\chi : G_K \to \mathbb{Z}_p^\times$ satisfy the Tate–Sen axioms (see [6, APPENDIX D] for a detailed exposition about Tate–Sen axioms) for any $c_1 > 0, c_2 > 0$ and $c_3 > \frac{1}{p-1}$ [4, Proposition 4.2.1, Proposition 3.1.4].

For any finite extension L of $\mathbb{Q}_p$, $n \in \mathbb{Z}$ and $s > 0$, let $\mathbf{A}_{L,n}^{\dagger,s}$ denote the subring $\varphi^{-n}(\mathbf{A}_L^{\dagger,p^n s})$ of $\widetilde{\mathbf{A}}^{\dagger,s}$ (see [2, §1.3] for the definitions of $\mathbf{A}_L^{\dagger,s}$ and $\widetilde{\mathbf{A}}^{\dagger,s}$). The following result then follows from [4, Proposition 3.3.1].

Proposition 2.1. *Let T_S be a free $\mathcal{O}_S$-linear representation of G_K of rank d. Let L be a finite Galois extension of K so that G_L acts trivially on $T_S/p^k T_S$, where k is an integer such that $\mathrm{val}^{(0,1]}(p^k) > c_1 + 2c_2 + 2c_3$. Then there exists an integer $n(L) \geq 0$ such that for any $n \geq n(L)$, $T_S \otimes_{\mathcal{O}_S} (\widetilde{\mathbf{A}}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S)$ has a unique sub-$\mathbf{A}_{L,n}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S$-module $\mathrm{D}_{L,n}^{\dagger,\frac{p-1}{p}}(T_S)$ which is free of rank d, fixed by H_L, stable under G_K, and has a basis which is c_3-fixed by Γ_L (that is, for each $\gamma \in \Gamma_L$, the matrix W_γ of γ with respect to this basis satisfies $\mathrm{val}^{(0,1]}(W_\gamma - 1) > c_3$), and satisfies*

$$\mathrm{D}_{L,n}^{\dagger,\frac{p-1}{p}}(T_S) \otimes_{\mathbf{A}_{L,n}^{\dagger,s} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S} (\widetilde{\mathbf{A}}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S) = T_S \otimes_{\mathcal{O}_S} (\widetilde{\mathbf{A}}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S). \quad (2.1)$$

Corollary 2.2. *Keep notations as above.*

(1) *We have* $\mathrm{D}_{L,n+1}^{\dagger,\frac{p-1}{p}}(T_S) = \mathrm{D}_{L,n}^{\dagger,s}(T_S) \otimes_{\mathbf{A}_{L,n}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S} (\mathbf{A}_{L,n+1}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S).$

(2) *By enlarging L and $n(L)$, we may have*

$$\mathrm{D}_{L,n+1}^{\dagger,\frac{p-1}{p}}(T_S) = \varphi^{-1}(\mathrm{D}_{L,n}^{\dagger,\frac{p-1}{p}}(T_S) \otimes_{\mathbf{A}_{L,n}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S} (\mathbf{A}_{L,n}^{\dagger,p-1} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S)).$$

Proof. Let D_1 and D_2 denote the right hand sides of (1) and (2) respectively. By (2.1), we first see that

$$D_1 \otimes_{\mathbf{A}_{L,n+1}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S} (\widetilde{\mathbf{A}}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S) = T_S \otimes_{\mathcal{O}_S} (\widetilde{\mathbf{A}}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S).$$

Moreover, it is straightforward to see that D_1 satisfies all the properties of $\mathrm{D}_{L,n+1}^{\dagger,\frac{p-1}{p}}(T_S)$ given by Proposition 2.1. We therefore conclude $D_1 = \mathrm{D}_{L,n+1}^{\dagger,\frac{p-1}{p}}(T_S)$ by the uniqueness of $\mathrm{D}_{L,n+1}^{\dagger,s}(T_S)$. This proves (1).

To prove (2), we enlarge L and $n(L)$ so that Proposition 2.1 holds both for (c_1, c_2, c_3) and for $(c_1, c_2, c_3' = pc_3)$. We first see that

$$\varphi(D_2) \otimes_{\mathbf{A}_{L,n}^{\dagger,p-1} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S} (\widetilde{\mathbf{A}}^{\dagger,p-1} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S) = T_S \otimes_{\mathcal{O}_S} (\widetilde{\mathbf{A}}^{\dagger,p-1} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S).$$

It follows that

$$D_2 \otimes_{\mathbf{A}_{L,n+1}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S} (\widetilde{\mathbf{A}}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S) = T_S \otimes_{\mathcal{O}_S} (\widetilde{\mathbf{A}}^{\dagger,\frac{p-1}{p}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S).$$

Let e be a c_3'-fixed basis of $\mathrm{D}_{L,n}^{\dagger,\frac{p-1}{p}}(T_S)$. It is then straightforward to see that the basis $\varphi^{-1}(e)$ of D_2 is at least c_3-fixed. We therefore conclude that $D_2 = \mathrm{D}_{L,n+1}^{\dagger,\frac{p-1}{p}}(T_S)$ by the uniqueness of $\mathrm{D}_{L,n+1}^{\dagger,\frac{p-1}{p}}(T_S)$. $\qquad\square$

In the rest of the paper, following the convention of [4], we fix some constants $c_1 > 0, c_2 > 0$ and $c_3 > \frac{p-1}{p}$ such that $c_1 + 2c_2 + 2c_3 < v_p(12p)$. Now let V_S be a free S-linear G_K-representation of rank d. Choose a free $\mathcal{O}_S$-lattice T_S in V_S. Since the G_K-action is continuous, there exists a finite Galois extension L of K such that G_L carries T_S into itself; hence T_S is G_L-stable. We may enlarge L so that G_L acts trivially on $T_S/12pT_S$. We also assume that Corollary 2.2(2) holds by further enlarging L and $n(L)$.

For any $g \in G_K$, it follows that gT_S is also a G_L-stable $\mathcal{O}_S$-lattice of V_S. Moreover, G_L acts trivially on $gT_S/12p(gT_S)$ as well. By the uniqueness of $\mathrm{D}_{L,n}^{\dagger,p-1/p}(gT_S)$, we get

$$\mathrm{D}_{L,n}^{\dagger,p-1/p}(gT_S) = g\mathrm{D}_{L,n}^{\dagger,p-1/p}(T_S).$$

Using the fact that T_S and gT_S are commensurable, we therefore deduce that $\mathrm{D}_{L,n}^{\dagger,p-1/p}(T_S)$ and $g\mathrm{D}_{L,n}^{\dagger,p-1/p}(T_S)$ are commensurable. This implies that the sub-S-module $\mathrm{D}_{L,n}^{\dagger,p-1/p}(T_S) \otimes_{\mathcal{O}_S} S$ of $V_S \otimes_{\mathcal{O}_S} (\widetilde{\mathbf{A}}^{\dagger,r_n} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S)$ is independent of the choice of T_S and G_K-stable for any $n \geq n(L)$. For $s \geq r_{n(L)}$, we set

$$\mathrm{D}_K^{\dagger,s}(V_S) = (\varphi^{n(L)}(\mathrm{D}_{L,n(L)}^{\dagger,p-1/p}(T_S)) \otimes_{\mathbf{A}_L^{\dagger,r_{n(L)}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S} \mathbf{B}_L^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S)^{H_K}$$

which is equipped with a Γ_K-action. By [4, Proposition 2.2.1] and [4, Lemme 4.2.5], there exists an $s(L/K) > 0$ such that if $s \geq s(L/K)$, then $\mathrm{D}_K^{\dagger,s}(V_S)$ is a locally free $\mathbf{B}_K^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S$-module of rank d. Let $n(V_S) = \max\{n(L), n(s(L/K))\}$, and put $s(V_S) = r_{n(V_S)}$.

Remark 2.3. By Corollary 2.2(2), we see that for any integers n_1, n_2 such that $n(L) \leq n_1, n_2 \leq n(s)$,

$$\varphi^{n_1}(\mathrm{D}_{L,n_1}^{\dagger,p-1/p}(T_S)) \otimes_{\mathbf{A}_L^{\dagger,r_{n_1}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S} \mathbf{B}_L^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S$$

$$= \varphi^{n_2}(\mathrm{D}_{L,n_2}^{\dagger,p-1/p}(T_S)) \otimes_{\mathbf{A}_L^{\dagger,r_{n_2}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S} \mathbf{B}_L^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S. \quad (2.2)$$

Thus one can replace $n(L)$ with any integer n such that $n(L) \leq n \leq n(s)$ in the construction of $\mathrm{D}_K^{\dagger,s}(V_S)$.

If $S \to R$ is a map of affinoid algebras over $\mathbb{Q}_p$, we set $V_R = V_S \otimes_S R$. The following theorem slightly refines [4, Théorème 4.2.9] in the case of affinoid algebras.

Theorem 2.4. *For any* $s \geq s(V_S)$, *the locally free* $\mathbf{B}_K^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S$-*module* $\mathrm{D}_K^{\dagger,s}(V_S)$ *is well-defined, i.e. its construction is independent of the choices of* T_S *and* L. *Furthermore, it satisfies the following properties.*

(1) *The natural map* $\mathrm{D}_K^{\dagger,s}(V_S) \otimes_{\mathbf{B}_K^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S} \widetilde{\mathbf{B}}_K^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S \rightarrow V_S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}_K^{\dagger,s}$ *is an isomorphism.*

(2) *The construction is compatible with base change in* S.

(3) *The construction is compatible with passage from* K *to a finite extension* L, *i.e.* $\mathrm{D}_L^{\dagger,s}(V_S) = \mathrm{D}_K^{\dagger,s}(V_S) \otimes_{\mathbf{B}_K^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S} \mathbf{B}_L^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S$.

(4) *For any* $s' \geq s$, $\mathrm{D}_K^{\dagger,s'}(V_S) = \mathrm{D}_K^{\dagger,s}(V_S) \otimes_{\mathbf{B}_K^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S} \mathbf{B}_K^{\dagger,s'} \widehat{\otimes}_{\mathbb{Q}_p} S$.

Proof. The statements (1) and (3) are already proved in [22, Theorem 3.11] (which in turn is an easy consequence of [4, Théorème 4.2.9]). The assertion (2) follows easily from the construction. For (4), let T_S and L be as above. It follows that

$$\mathrm{D}_L^{\dagger,s}(V_S) = \varphi^{n(L)}(\mathrm{D}_{L,n(L)}^{\dagger,p^{-1/p}}(T_S)) \otimes_{\mathbf{A}_L^{\dagger,r_{n(L)}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S} \mathbf{B}_L^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S$$

for any $s \geq s(V_S)$. This implies

$$\mathrm{D}_L^{\dagger,s'}(V_S) = \mathrm{D}_L^{\dagger,s}(V_S) \otimes_{\mathbf{B}_L^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S} \mathbf{B}_L^{\dagger,s'} \widehat{\otimes}_{\mathbb{Q}_p} S.$$

By (3), we get

$$\mathrm{D}_K^{\dagger,s'}(V_S) \otimes_{\mathbf{B}_K^{\dagger,s'} \widehat{\otimes}_{\mathbb{Q}_p} S} \mathbf{B}_L^{\dagger,s'} \widehat{\otimes}_{\mathbb{Q}_p} S = \mathrm{D}_K^{\dagger,s}(V_S) \otimes_{\mathbf{B}_K^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S} \mathbf{B}_L^{\dagger,s'} \widehat{\otimes}_{\mathbb{Q}_p} S.$$

We conclude by taking the H_K-invariants on both sides. $\square$

From now on, we assume $s \geq s(V_S)$ unless specified otherwise.

Proposition 2.5. *We have* $\varphi(\mathrm{D}_K^{\dagger,s}(V_S)) \subset \mathrm{D}_K^{\dagger,ps}(V_S)$ *and the natural map*

$$\varphi(\mathrm{D}_K^{\dagger,s}(V_S)) \otimes_{\mathbf{B}_K^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S, \varphi} \mathbf{B}_K^{\dagger,ps} \widehat{\otimes}_{\mathbb{Q}_p} S \rightarrow \mathrm{D}_K^{\dagger,ps}(V_S)$$

is an isomorphism.

Proof. Let T_S be a free $\mathcal{O}_S$-lattice of S, and let L be a finite Galois extension of K such that T_S is G_L-stable, and G_L acts trivially on $T_S/12pT_S$. By Corollary (2.2)(1), we get

$$\varphi(\mathrm{D}_L^{\dagger,s}(V_S)) \otimes_{\mathbf{B}_L^{\dagger,s}\widehat{\otimes}_{\mathbb{Q}_p}S,\varphi} (\mathbf{B}_L^{\dagger,ps}\widehat{\otimes}_{\mathbb{Q}_p}S)$$

$$=\varphi(\varphi^{n(L)}(\mathrm{D}_{L,n(L)}^{\dagger,\frac{p-1}{p}}(T_S)) \otimes_{\mathbf{A}_L^{\dagger,r_{n(L)}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_S} (\mathbf{B}_L^{\dagger,s}\widehat{\otimes}_{\mathbb{Q}_p}S))$$

$$\otimes_{\mathbf{B}_L^{\dagger,s}\widehat{\otimes}_{\mathbb{Q}_p}S,\varphi} \mathbf{B}_L^{\dagger,ps}\widehat{\otimes}_{\mathbb{Q}_p}S$$

$$=\varphi^{n(L)+1}(\mathrm{D}_{L,n(L)}^{\dagger,\frac{p-1}{p}}(T_S)) \otimes_{\mathbf{A}_L^{\dagger,r_{n(L)}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_S,\varphi} \mathbf{B}_L^{\dagger,ps}\widehat{\otimes}_{\mathbb{Q}_p}S$$

$$=\varphi^{n(L)+1}(\mathrm{D}_{L,n(L)}^{\dagger,\frac{p-1}{p}}(T_S)) \otimes_{\mathbf{A}_L^{\dagger,r_{n(L)}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_S,\varphi} (\mathbf{A}_L^{\dagger,r_{n(L)+1}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_S)$$

$$\otimes_{\mathbf{A}_L^{\dagger,r_{n(L)+1}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_S} \mathbf{B}_L^{\dagger,ps}\widehat{\otimes}_{\mathbb{Q}_p}S$$

$$=\varphi^{n(L)+1}(\mathrm{D}_{L,n(L)}^{\dagger,\frac{p-1}{p}}(T_S) \otimes_{\mathbf{A}_{L,n(L)}^{\dagger,r_{n(L)}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_S} \mathbf{A}_{L,n(L)+1}^{\dagger,r_{n(L)+1}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_S)$$

$$\otimes_{\mathbf{A}_L^{\dagger,r_{n(L)+1}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_S} \mathbf{B}_L^{\dagger,ps}\widehat{\otimes}_{\mathbb{Q}_p}S$$

$$=\varphi^{n(L)+1}(\mathrm{D}_{L,n(L)+1}^{\dagger,\frac{p-1}{p}}(T_S)) \otimes_{\mathbf{A}_L^{\dagger,r_{n(L)+1}}\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_S} \mathbf{B}_L^{\dagger,ps}\widehat{\otimes}_{\mathbb{Q}_p}S$$

$$=\mathrm{D}_L^{\dagger,ps}(V_S).$$

The last step follows from Remark 2.3. By Theorem 2.4(3), we may rewrite the above equality as

$$\varphi(\mathrm{D}_K^{\dagger,s}(V_S)) \otimes_{\mathbf{B}_K^{\dagger,s}\widehat{\otimes}_{\mathbb{Q}_p}S,\varphi} \mathbf{B}_L^{\dagger,ps}\widehat{\otimes}_{\mathbb{Q}_p}S = \mathrm{D}_K^{\dagger,ps}(V_S) \otimes_{\mathbf{B}_K^{\dagger,ps}\widehat{\otimes}_{\mathbb{Q}_p}S} \mathbf{B}_L^{\dagger,ps}\widehat{\otimes}_{\mathbb{Q}_p}S.$$

We conclude by taking H_K-invariants on both sides. $\square$

We set $\mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_S) = \mathrm{D}_K^{\dagger,s}(V_S) \otimes_{\mathbf{B}_K^{\dagger,s}\widehat{\otimes}_{\mathbb{Q}_p}S} (\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}\widehat{\otimes}_{\mathbb{Q}_p}S)$. We put

$$\mathbf{B}_K^{\dagger}\widehat{\otimes}_{\mathbb{Q}_p}S = \cup_{s>0}\mathbf{B}_K^{\dagger,s}\widehat{\otimes}_{\mathbb{Q}_p}S, \quad \widetilde{\mathbf{B}}_K^{\dagger}\widehat{\otimes}_{\mathbb{Q}_p}S = \cup_{s>0}\widetilde{\mathbf{B}}_K^{\dagger,s}\widehat{\otimes}_{\mathbb{Q}_p}S$$

and

$$\mathbf{B}_{\mathrm{rig},K}^{\dagger}\widehat{\otimes}_{\mathbb{Q}_p}S = \cup_{s>0}\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}\widehat{\otimes}_{\mathbb{Q}_p}S, \quad \widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger}\widehat{\otimes}_{\mathbb{Q}_p}S = \cup_{s>0}\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger,s}\widehat{\otimes}_{\mathbb{Q}_p}S.$$

We then set

$$\mathrm{D}_K^{\dagger}(V_S) = \mathrm{D}_K^{\dagger,s}(V_S) \otimes_{\mathbf{B}_K^{\dagger,s}\widehat{\otimes}_{\mathbb{Q}_p}S} \mathbf{B}_K^{\dagger}\widehat{\otimes}_{\mathbb{Q}_p}S = \cup_{s\geq s(V_S)}\mathrm{D}_K^{\dagger,s}(V_S)$$

and

$$\mathrm{D}_{\mathrm{rig},K}^{\dagger}(V_S) = \mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_S) \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}\widehat{\otimes}_{\mathbb{Q}_p}S} \mathbf{B}_{\mathrm{rig},K}^{\dagger}\widehat{\otimes}_{\mathbb{Q}_p}S = \cup_{s\geq s(V_S)}\mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_S).$$

By Proposition 2.5, we see that $D_K^\dagger(V_S)$ and $D_{\mathrm{rig},K}^\dagger(V_S)$ are stable under φ and isomorphic to their φ-pullbacks respectively. That is, the natural morphisms $\varphi^*(D_K^\dagger(V_S)) \to D_K^\dagger(V_S)$ and $\varphi^*(D_{\mathrm{rig},K}^\dagger(V_S)) \to D_{\mathrm{rig},K}^\dagger(V_S)$ are isomorphisms. Thus $D_K^\dagger(V_S)$ is a (φ, Γ)-module over $\mathbf{B}_K^\dagger \widehat{\otimes}_{\mathbb{Q}_p} S$ in the sense of [22]. (See Remark 5.21 for the relevant discussion about $D_{\mathrm{rig},K}^\dagger(V_S)$.)

Remark 2.6. In the case when V_S admits a G_K-stable free $\mathcal{O}_S$-lattice T_S, we further have that the (φ, Γ)-module $D_K^\dagger(V_S)$ is *globally étale* in the sense of [27]. In fact, if L is a finite Galois extension of K so that G_L acts trivially on $T_S/12pT_S$,

$$\mathbf{A}_K^\dagger(T_S) = \cup_{s \geq s(V_S)} (\varphi^{n(L)}(D_{L,n(L)}^{\dagger,\frac{p-1}{p}}(T_S)) \otimes_{\mathbf{A}^{\dagger,r_{n(L)}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S} \mathbf{A}_L^{\dagger,s} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S)^{H_K}$$

is a locally free $\mathbf{A}_L^\dagger \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_S$-lattice of $D_K^\dagger(V_S)$ which satisfies

$$\varphi^*(\mathbf{A}_K^\dagger(T_S)) = \mathbf{A}_K^\dagger(T_S).$$

Corollary 2.7. *Let $a \in D_{\mathrm{rig},K}^{\dagger,ps}(V_S)$. If $\varphi(a) \in D_{\mathrm{rig},K}^{\dagger,ps}(V_S)$, then $a \in D_{\mathrm{rig},K}^{\dagger,s}(V_S)$.*

Proof. Let T_S be a free $\mathcal{O}_S$-lattice of V_S, and let L be a finite Galois extension of K so that G_L acts trivially on $T_S/12pT_S$. By its construction $D_L^{\dagger,s}(V_S)$ is a free $\mathbf{B}_L^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S$-module of rank d. Let $e_1, \ldots, e_d$ be a basis, and write $a = \sum_{i=1}^d a_i e_i$ with $a_i \in \mathbf{B}_{\mathrm{rig},L}^{\dagger,ps} \widehat{\otimes}_{\mathbb{Q}_p} S$. Since $D_{\mathrm{rig},K}^{\dagger,s}(V_S) = (D_{\mathrm{rig},L}^{\dagger,s}(V_S))^{H_K}$, it reduces to show that $a \in D_{\mathrm{rig},L}^{\dagger,s}(V_S)$. By Proposition 2.5, $\varphi(e_1), \ldots, \varphi(e_d)$ form a $\mathbf{B}_{\mathrm{rig},L}^{\dagger,ps} \widehat{\otimes}_{\mathbb{Q}_p} S$-basis of $D_L^{\dagger,s}(V_S)$. Hence $\varphi(a) = \sum_{i=1}^d \varphi(a_i)\varphi(e_i)$ belongs to $D_{\mathrm{rig},L}^{\dagger,ps}(V_S)$ if and only if $\varphi(a_i) \in \mathbf{B}_{\mathrm{rig},L}^{\dagger,ps} \widehat{\otimes}_{\mathbb{Q}_p} S$ for all i. The latter is equivalent to $a_i \in \mathbf{B}_{\mathrm{rig},L}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S$ for all i. This yields the desired result. $\square$

2.2. Sheafification of the (φ, Γ)-module functor. Following [22], we extend the (φ, Γ)-module functors to finite locally free S-linear representations as follows. From now on, let V_S be a locally free S-linear representation of G_K of rank d. We choose a finite covering of $M(S)$ by affinoid subdomains $M(S_1), \ldots, M(S_m)$ such that V_{S_i} is free over S_i for each i. Let $s_0 = \max_{1 \leq i \leq m}\{s(V_{S_i})\}$. By [22, Lemma 3.3] and Theorem 2.4, for any $s \geq s_0$ and $1 \leq i \leq m$, the presheaf $\mathscr{D}_K^{\dagger,s}(V_{S_i}) : R_i \mapsto D_K^{\dagger,s}(V_{R_i})$, where $M(R_i)$ runs through all affinoid subdomains of $M(S_i)$, is indeed a sheaf for the weak G-topology of $M(S_i)$ (hence extends uniquely to the strong G-topology). We glue the sheaves $\mathscr{D}_K^{\dagger,s}(V_{S_i})$ for all $1 \leq i \leq m$ to form a sheaf $\mathscr{D}_K^{\dagger,s}(V_S)$ on $M(S)$, which is independent of the choice of the covering. It turns out that $\mathscr{D}_K^{\dagger,s}(V_S)$ is the sheaf associated to a finite locally free $\mathbf{B}_K^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S$-module $D_K^{\dagger,s}(V_S)$ [22, Proposition 3.6]. It is straightforward to see that $D_K^{\dagger,s}(V_S)$ is equipped

with a natural Γ-action, and the construction $V_S \mapsto \mathrm{D}_K^{\dagger,s}(V_S)$ satisfies the analogues of the assertions of Theorem 2.4.

We then define the functors $\mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_S)$, $\mathrm{D}_K^{\dagger}(V_S)$ and $\mathrm{D}_{\mathrm{rig},K}^{\dagger}(V_S)$ as in §2.1. The sheaf property for $\mathscr{D}_K^{\dagger,s}(V_S)$ and Proposition 2.5 ensure that $\mathrm{D}_K^{\dagger}(V_S)$ is isomorphic to its φ-pullback. Hence $\mathrm{D}_K^{\dagger}(V_S)$ and $\mathrm{D}_{\mathrm{rig},K}^{\dagger}(V_S)$ are (φ, Γ)-modules over $\mathbf{B}_K^{\dagger} \widehat{\otimes}_{\mathbb{Q}_p} S$ and $\mathbf{B}_{\mathrm{rig},K}^{\dagger} \widehat{\otimes}_{\mathbb{Q}_p} S$ respectively.

Note that for any affinoid subdomain $M(S')$ of $M(S)$, one may proceed the above constructions for $V_{S'}$ using the covering $M(S_1) \cap M(S'), \ldots, M(S_m) \cap M(S')$. In particular, one can define $\mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_{S'})$ for any $s \geq s_0$.

Definition 2.8. For any $s \geq s_0$, define the presheaves $\mathscr{D}_{\mathrm{rig},K}^{\dagger,s}(V_S)$ and $\mathscr{D}_{\mathrm{rig},K}^{\dagger}(V_S)$ on the weak G-topology of $M(S)$ by setting

$$\mathscr{D}_{\mathrm{rig},K}^{\dagger,s}(V_S)(M(S')) = \mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_{S'}), \quad \mathscr{D}_{\mathrm{rig},K}^{\dagger}(V_S)(M(S')) = \mathrm{D}_{\mathrm{rig},K}^{\dagger}(V_{S'})$$

for any affinoid subdomain $M(S')$ of $M(S)$.

Proposition 2.9. *Both $\mathscr{D}_{\mathrm{rig},K}^{\dagger,s}(V_S)$ and $\mathscr{D}_{\mathrm{rig},K}^{\dagger}(V_S)$ are sheaves for the weak G-topology of $M(S)$, and hence extend uniquely to the strong G-topology.*

Proof. We first show that $\mathscr{D}_{\mathrm{rig},K}^{\dagger,s}(V_S)$ is a sheaf. This amounts to checking the sheaf condition for finite coverings of affinoid subdomains by affinoids. Recall that

$$\mathscr{D}_{\mathrm{rig},K}^{\dagger,s}(V_S)(M(S')) = \mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_{S'}) = \mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_S) \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S} \mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S'$$

for any affinoid subdomain $M(S')$ of $M(S)$. Since $\mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_S)$ is a finite locally free $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S$-module, it reduces to show that the presheaf $M(S') \mapsto \mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S'$ is a sheaf on the weak G-topology of $M(S)$. By definition, $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ is the Fréchet completion of $\mathbf{B}_K^{\dagger,s}$ with respect to the set of valuations $\{\mathrm{val}^{(0,r]}\}_{r \geq s}$. For $r \geq s$, let $\mathbf{B}_K^{[s,r]}$ be the completion of $\mathbf{B}_K^{\dagger,s}$ with respect to $\max\{\mathrm{val}^{(0,r]}, \mathrm{val}^{(0,s]}\}$. It follows that

$$\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} = \varprojlim_r \mathbf{B}_K^{[s,r]}.$$

Using a Schauder basis of S, we deduce

$$\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S = \varprojlim_r \mathbf{B}_K^{[s,r]} \widehat{\otimes}_{\mathbb{Q}_p} S. \tag{2.3}$$

Therefore, it suffices to show that the presheaf defined by $M(S') \mapsto \mathbf{B}_K^{[s,r]} \widehat{\otimes}_{\mathbb{Q}_p} S'$ is a sheaf on the weak G-topology of $M(S)$; this follows from [22, Lemma 3.3].

Note that the presheaf $\mathscr{D}_{\mathrm{rig},K}^{\dagger}(V_S)$ is the direct limit of the sheaves $\mathscr{D}_{\mathrm{rig},K}^{\dagger,s}(V_S)$ in the category of presheaves. In general, the direct limit of sheaves in the category

of presheaves is not necessarily a sheaf. However, using the facts that the coverings are all finite and the connecting maps $\mathscr{D}_{\mathrm{rig},K}^{\dagger,s_1}(V_S) \to \mathscr{D}_{\mathrm{rig},K}^{\dagger,s_2}(V_S)$ are injective for all $s_1 < s_2$, it is straightforward to check that the direct limit of the sheaves $\mathscr{D}_{\mathrm{rig},K}^{\dagger,s}(V_S)$ in the category of presheaves is indeed a sheaf. $\qquad\square$

Theorem 2.10. *The constructions* $\mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_S)$ *and* $\mathrm{D}_{\mathrm{rig},K}^{\dagger}(V_S)$ *for finite locally free S-linear representations* V_S *have the same properties as for finite free S-linear representations given in §2.1.*

Proof. We choose a finite covering of $M(S)$ by affinoid subdomains such that the restriction of V_S on each piece is free. The theorem then follows from Proposition 2.9. $\qquad\square$

The following lemma will be used in §4.

Lemma 2.11. *Let* $a \in \mathrm{D}_{\mathrm{rig},K}^{\dagger}(V_S)$ *and* $\alpha \in S$. *If* $\varphi^m(a) - \alpha a \in \mathrm{D}_{\mathrm{rig},K}^{\dagger,p^m s}(V_S)$, *then* $a \in \mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_S)$.

Proof. Put $b = \varphi^m(a) - \alpha a$. Suppose that $a \in \mathrm{D}_{\mathrm{rig},K}^{\dagger,s'}(V_S)$ for some s'. If $s' > s$, we get $\varphi^m(a) = b + \alpha a \in \mathrm{D}_{\mathrm{rig},K}^{\dagger,s'}(V_S)$. It follows from Corollary 2.7 and Theorem 2.10 that $a \in \mathrm{D}_{\mathrm{rig},K}^{\dagger,s''}(V_S)$ for $s'' = \max\{s'/p^m, s\}$. We then conclude $a \in \mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_S)$ by iterating this argument. $\qquad\square$

Definition 2.12. Let X be a rigid analytic space over $\mathbb{Q}_p$, and let V_X be a locally free coherent $\mathcal{O}_X$-module equipped with a continuous $\mathcal{O}_X$-linear G_K-action. We choose an admissible covering of X by affinoid subdomains $\{M(S_i)\}_{i \in I}$. We then define the sheaf $\mathscr{D}_{\mathrm{rig},K}^{\dagger}(V_X)$ by gluing the sheaves $\mathscr{D}_{\mathrm{rig},K}^{\dagger}(V_{S_i})$ for all $i \in I$; this construction is independent of the choice of the covering $\{M(S_i)\}_{i \in I}$.

2.3. Localization maps. Recall that Fontaine's p-adic $2\pi i$ is defined as $t = \log[\varepsilon]$. We equip $K_n[[t]]$ with the induced Fréchet topology via the natural identification $K_n[[t]] \cong K_n^{\mathbb{N}}$. We define $K_n((t)) \widehat{\otimes}_{\mathbb{Q}_p} S$ as the inductive limit of $(t^{-i} K_n[[t]]) \widehat{\otimes}_{\mathbb{Q}_p} S$. Recall that for any $n \geq n(s)$, there is a continuous Γ-equivariant injective map

$$\iota_n : \mathbf{B}_K^{\dagger,s} \to K_n[[t]],$$

which extends to a continuous Γ-equivariant injective map

$$\iota_n : \mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \to K_n[[t]].$$

It is defined as the composite

$$\mathbf{B}_K^{\dagger,s} \subset \widetilde{\mathbf{B}}^{\dagger,s} \xrightarrow{\varphi^{-n}} \widetilde{\mathbf{B}}^{\dagger,p^{-n}s} \subset \widetilde{\mathbf{B}}^+ \subset \mathbf{B}_{\mathrm{dR}}^+.$$

and it factors through $K_n[[t]]$ (see [2, §2] for more details about ι_n). In particular we have $\iota_{n+1} \circ \varphi = \iota_n$. The map ι_n induces a continuous Γ-equivariant map

$$\iota_n : \mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S \to K_n[[t]] \widehat{\otimes}_{\mathbb{Q}_p} S.$$

We define

$$\mathrm{D}_{\mathrm{dif}}^{+,K_n}(V_S) = \mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_S) \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S, \iota_n} (K_n[[t]] \widehat{\otimes}_{\mathbb{Q}_p} S)$$

and

$$\mathrm{D}_{\mathrm{dif}}^{K_n}(V_S) = \mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_S) \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S, \iota_n} (K_n((t)) \widehat{\otimes}_{\mathbb{Q}_p} S);$$

it is clear that $\mathrm{D}_{\mathrm{dif}}^{K_n}(V_S) = \mathrm{D}_{\mathrm{dif}}^{+,K_n}(V_S)[1/t]$. We denote by ι_n the natural map

$$\mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_S) \to \mathrm{D}_{\mathrm{dif}}^{+,K_n}(V_S),$$

and call it the *localization map*. It is straightforward to see that $\varphi : \mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \to \mathbf{B}_{\mathrm{rig},K}^{\dagger,ps}$ induces a $K_n[[t]]$-linear morphism $\mathrm{D}_{\mathrm{dif}}^{+,K_n}(V_S) \to \mathrm{D}_{\mathrm{dif}}^{+,K_{n+1}}(V_S)$ which induces an isomorphism

$$\mathrm{D}_{\mathrm{dif}}^{+,K_n}(V_S) \otimes_{K_n[[t]]} K_{n+1}[[t]] \cong \mathrm{D}_{\mathrm{dif}}^{+,K_{n+1}}(V_S).$$

We define $\mathrm{D}_{\mathrm{Sen}}^{K_n}(V_S) = \mathrm{D}_{\mathrm{dif}}^{+,K_n}(V_S)/(t)$. Finally, we define

$$\mathrm{D}_{\mathrm{dif}}^{+,K_n}(V_S) = \cup_{n \geq n(s)} \mathrm{D}_{\mathrm{dif}}^{+,K_n}(V_S), \quad \mathrm{D}_{\mathrm{dif}}^{K}(V_S) = \cup_{n \geq n(s)} \mathrm{D}_{\mathrm{dif}}^{K_n}(V_S),$$

$$\mathrm{D}_{\mathrm{Sen}}^{K}(V_S) = \cup_{n \geq n(s)} \mathrm{D}_{\mathrm{Sen}}^{K_n}(V_S).$$

Convention 2.13. When the base field K is clear, we omit K in all of these functors for simplicity.

By their constructions and the base change properties of (φ, Γ)-module functors, the following proposition is obvious.

Proposition 2.14. *The functors $\mathrm{D}_{\mathrm{dif}}^{+,n}$ and $\mathrm{D}_{\mathrm{Sen}}^{n}$ are compatible with base change.*

Let $q = \varphi([\varepsilon] - 1)/([\varepsilon] - 1)$. The following proposition is a generalization of [26, Theorem 4.3].

Proposition 2.15. *Let k be a positive integer. The following are true.*

(1) *The localization map $\iota_n : \mathrm{D}_{\mathrm{rig}}^{\dagger,s}(V_S) \to \mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)$ induces an isomorphism*

$$\mathrm{D}_{\mathrm{rig}}^{\dagger,s}(V_S)/(\varphi^{n-1}(q))^k \cong \mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k).$$

(2) *The natural map $\prod_{n \geq n(s)} \iota_n : \mathrm{D}_{\mathrm{rig}}^{\dagger,s}(V_S) \to \prod_{n \geq n(s)} \mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)$ induces an isomorphism*

$$\mathrm{D}_{\mathrm{rig}}^{\dagger,s}(V_S)/(t^k) \cong \prod_{n \geq n(s)} \mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k).$$

(3) *The natural map* φ : $\mathrm{D}_{\mathrm{rig}}^{\dagger,s}(V_S)/(t^k) \rightarrow \mathrm{D}_{\mathrm{rig}}^{\dagger,ps}(V_S)/(t^k)$ *is given by* $((a_n)_{n \geq n(s)}) \rightarrow ((a_{n-1})_{n \geq n(s)+1})$ *under the isomorphism of* (2),

Proof. For (1) and (2), since $\mathrm{D}_{\mathrm{rig}}^{\dagger,s}(V_S)$ is a finite locally free $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S$-module, it reduces to show that

$$(\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S)/(\varphi^{n-1}(q))^k \cong (K_n[[t]]\widehat{\otimes}_{\mathbb{Q}_p} S)/(t^k) \qquad (2.4)$$

and

$$(\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S)/(t^k) \cong \prod_{n \geq n(s)} (K_n[[t]]\widehat{\otimes}_{\mathbb{Q}_p} S)/(t^k). \qquad (2.5)$$

We first show them for $S = \mathbb{Q}_p$. By [2, Proposition 4.8], for $f \in \mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$, $t|\iota_n(f)$ if and only if $\varphi^{n-1}(q)|f$. Note that $t||\iota_n(\varphi^{n-1}(q))$. We thus deduce that the map

$$\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}/(\varphi^{n-1}(q))^k \rightarrow K_n[[t]]/(t^k)$$

is injective. Furthermore, it is an isomorphism for $k = 1$ by [2, lemme 4.9]. It follows that it is an isomorphism for any k. Since $t = \prod_{n \geq n(s)}(\varphi^{n-1}(q)/p)$ in $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$, we further get

$$\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}/(t^k) \cong \prod_{n \geq n(s)} \mathbf{B}_{\mathrm{rig},K}^{\dagger,s}/\varphi^{n-1}(q) \cong \prod_{n \geq n(s)} K_n[[t]]/(t^k).$$

We claim that the exact sequence

$$0 \rightarrow (\varphi^{n-1}(q))^k \mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \rightarrow \mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \rightarrow \mathbf{B}_{\mathrm{rig},K}^{\dagger,s}/(\varphi^{n-1}(q))^k \rightarrow 0$$

splits as complete Fréchet spaces over $\mathbb{Q}_p$. Note that the quotient $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}/(\varphi^{n-1}(q))^k$ is a finite dimensional $\mathbb{Q}_p$-vector space. We choose a section of the $\mathbb{Q}_p$-linear map $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \rightarrow \mathbf{B}_{\mathrm{rig},K}^{\dagger,s}/(\varphi^{n-1}(q))^k$, and denote its image by U. Note that for any $r \geq s$, $\mathrm{val}^{(0,r]}$ is indeed a norm on $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$. Since every finite dimensional normed $\mathbb{Q}_p$-vector space is complete, we deduce that U is a closed Fréchet subspace of $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$. By open mapping theorem for Fréchet spaces [32, Proposition 8.6], the map $(\varphi^{n-1}(q))^k \mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \oplus U \rightarrow \mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ is an isomorphism of $\mathbb{Q}_p$-Fréchet spaces. This proves the claim.

The claim yields the following exact sequence

$$0 \rightarrow (\varphi^{n-1}(q))^k \mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S \rightarrow \mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S \rightarrow (\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}/(\varphi^{n-1}(q))^k)\widehat{\otimes}_{\mathbb{Q}_p} S \rightarrow 0;$$

hence

$$(\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S)/(\varphi^{n-1}(q))^k \cong (\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}/(\varphi^{n-1}(q))^k)\widehat{\otimes}_{\mathbb{Q}_p} S$$

$$\cong K_n[[t]]/(t^k) \otimes_{\mathbb{Q}_p} S \cong (K_n[[t]]\widehat{\otimes}_{\mathbb{Q}_p} S)/(t^k),$$

yielding (2.4). We get (2.5) by a similar argument. We get (3) immediately from the fact that $\iota_{n+1} \circ \varphi = \iota_n$ for all $n \geq n(s)$. $\qquad\square$

Note that φ^f acts K_0-linearly on $D_{rig}^\dagger(V_S)$. We extend the φ^f-action to $K \otimes_{K_0} D_{rig}^\dagger(V_S)$ K-linearly. For $s \geq s(V_S)$ and $n \geq \max\{n(V_S), n(s)/f\}$, we set

$$\iota_{n,K} : K \otimes_{K_0} D_{rig}^{\dagger,s}(V_S) \to D_{dif}^{+,f n}(V_S)$$

as the K-linear extension of ι_{fn}. Recall that every closed ideal of $\mathbf{B}_{rig,K}^{\dagger,s}$ is principal [20, Theorem 2.9.6]. It follows that every closed ideal of $K \otimes_{K_0} \mathbf{B}_{rig,K}^{\dagger,s}$ is principal. Thus the closed ideal $\iota_{n,K}^{-1}((t))$ of $K \otimes_{K_0} \mathbf{B}_{rig,K}^{\dagger,s}$ is principal; we fix a generator $q_{n,K}$ of it. It follows that $\iota_{n,K}$ induces an isomorphism $(K \otimes_{K_0} \mathbf{B}_{rig,K}^{\dagger,s})/(q_{n,K}^k) \cong K_n[[t]]/(t^k)$. Again, the closed ideal $\cap_n (q_{n,K})$ is principal; we fix a generator t_K of it.

Proposition 2.16. *The following are true.*

(1) *The ideal $(q_{n,K})$ is a prime factor of $(\varphi^{f n-1}(q))$.*

(2) *The map $\iota_{n,K}$ induces an isomorphism*

$$(K \otimes_{K_0} D_{rig}^{\dagger,s}(V_S))/(q_{n,K}^k) \cong D_{dif}^{+,f n}(V_S)/(t^k)$$

for any $k \geq 1$ and $s \leq r_{fn}$.

(3) *We have $\varphi^f((q_{n,K})) = (q_{n+1,K})$.*

(4) *For any $k \geq 1$, the natural map*

$$\prod_{fn \geq n(s)} \iota_{n,K} : K \otimes_{K_0} D_{rig}^{\dagger,s}(V_S) \to \prod_{fn \geq n(s)} D_{dif}^{+,f n}(V_S)$$

induces an isomorphism

$$(K \otimes_{K_0} D_{rig}^{\dagger,s}(V_S))/(t_K^k) \cong \prod_{fn \geq n(s)} D_{dif}^{+,f n}(V_S)/(t^k).$$

(5) *The valuation of $\varphi^f(t_K)/t_K$, which is viewed as an element of $K \otimes_{K_0} \mathbf{B}_K^\dagger$, is 1; here we put the valuation of π_K in $K \otimes_{K_0} \mathbf{B}_K^\dagger$ to be 1.*

Proof. By Proposition 2.15(1), ι_{fn} induces an isomorphism

$$(K \otimes_{K_0} D_{rig}^{\dagger,s}(V_S))/(\varphi^{f n-1}(q)^k) \cong K \otimes_{K_0} D_{dif}^{+,f n}(V_S)/(t^k).$$

The map $K \otimes_{K_0} D_{rig}^{\dagger,s}(V_S) \to D_{dif}^{+,f n}(V_S)/(t^k)$ is then just the composite

$$K \otimes_{K_0} D_{rig}^{\dagger,s}(V_S) \to (K \otimes_{K_0} D_{rig}^{\dagger,s}(V_S))/(\varphi^{f n-1}(q)^k)$$

$$\cong K \otimes_{K_0} D_{dif}^{+,f n}(V_S)/(t^k) \to D_{dif}^{+,f n}(V_S)/(t^k).$$

This implies the first two statements. We deduce (3) from Proposition 2.15(3). Note that the ideal (t_K) is the product of all $(q_{n,K})$ which are mutually prime by (1). We then deduce (4) using a similar argument as in the proof of Proposition 2.15(2). For (5), we choose a generator $t_f \in \mathbf{B}^{\dagger,s}_{\mathrm{rig},K}$ of the closed ideal $\cap_n(\varphi^{fn-1}(q))$. By Proposition 2.15(1), (2), it is straightforward to see that

$$\prod_{i=0}^{f-1} \varphi^i((t_f)) = (t).$$

This yields that $\varphi^f(t_f)/t_f$ belongs to $(\mathbf{B}^{\dagger,s}_{\mathrm{rig},K})^\times \subset \mathbf{B}^\dagger_K$ and has valuation 1 as $\varphi(t) = pt$; hence its valuation in $K \otimes_{K_0} \mathbf{B}^\dagger_K$ is equal to the ramification index $e = [K : \mathbb{Q}_p]/f$. After a suitable base change, we may assume that K is Galois over K_0. A short computation shows that

$$\prod_{\sigma \in \mathrm{Gal}(K/K_0)} \sigma((t_K)) = (t_f)$$

as ideals of $K \otimes_{K_0} \mathbf{B}^{\dagger,s}_{\mathrm{rig},K}$. It follows that

$$\varphi^f(\sigma(t_K))/\sigma(t_K) \in (K \otimes_{K_0} \mathbf{B}^\dagger_{\mathrm{rig},K})^\times \subset K \otimes_{K_0} \mathbf{B}^\dagger_K$$

for each $\sigma \in \mathrm{Gal}(K/K_0)$, and their product is $\varphi^f(t_f)/t_f$ up to a unit of $K \otimes_{K_0} \mathbf{B}^\dagger_K$. Since they all have the same valuation, we conclude (5) by the fact that $\varphi^f(t_f)/t_f$ has valuation e in $K \otimes_{K_0} \mathbf{B}^\dagger_K$. $\quad\square$

2.4. The sheaf $(\mathscr{D}^{+,n}_{\mathrm{dif}}(V_S)/(t^k))^\Gamma$.

Definition 2.17. Define the presheaves $\mathscr{D}^{+,n}_{\mathrm{dif}}(V_S)$ and $\mathscr{D}^{+,n}_{\mathrm{dif}}(V_S)/(t^k)$ on the weak G-topology of $M(S)$ by setting

$$(\mathscr{D}^{+,n}_{\mathrm{dif}}(V_S))(M(S')) = \mathrm{D}^{+,n}_{\mathrm{dif}}(V_{S'}), \quad (\mathscr{D}^{+,n}_{\mathrm{dif}}(V_S)/(t^k))(M(S')) = \mathrm{D}^{+,n}_{\mathrm{dif}}(V_{S'})/(t^k)$$

for any affinoid subdomain $M(S')$ of $M(S)$. Define the presheaves

$$\mathscr{D}^+_{\mathrm{dif}}(V_S) = \varinjlim_{n\to\infty} \mathscr{D}^{+,n}_{\mathrm{dif}}(V_S) \quad \text{and} \quad \mathscr{D}^+_{\mathrm{dif}}(V_S)/(t^k) = \varinjlim_{n\to\infty} \mathscr{D}^{+,n}_{\mathrm{dif}}(V_S)/(t^k),$$

where the transition maps on any affinoid subdomain $M(S')$ of $M(S)$ are the natural $K_n[[t]]$-linear morphisms $\mathrm{D}^{+,n}_{\mathrm{dif}}(V_{S'}) \to \mathrm{D}^{+,n+1}_{\mathrm{dif}}(V_{S'})$ and $\mathrm{D}^{+,n}_{\mathrm{dif}}(V_{S'})/(t^k) \to \mathrm{D}^{+,n+1}_{\mathrm{dif}}(V_{S'})/(t^k)$ introduced in §2.3.

Proposition 2.18. *The presheaves $\mathscr{D}^{+,n}_{\mathrm{dif}}(V_S)$ and $\mathscr{D}^+_{\mathrm{dif}}(V_S)$ are sheaves for the weak G-topology of $M(S)$, and hence extend uniquely to the strong G-topology.*

Proof. As in the proof of Proposition 2.9, by the base change property of the functor $D_{dif}^{+,n}$, it reduces to show that the presheaf

$$M(S') \mapsto K_n[[t]] \widehat{\otimes}_{\mathbb{Q}_p} S'$$

is a sheaf. The latter is clear. $\square$

Lemma 2.19. *Let G be a topologically finitely generated group. Let A be a commutative Hausdoff topological ring, and let M be a finite A-module equipped with a continuous A-linear action of G. Suppose B is a commutative Hausdoff topological ring with a continuous flat morphism $A \to B$. Then $(M \otimes_A B)^G = M^G \otimes_A B$.*

Proof. Choose a finite set of topological generators $g_1, \dots, g_n$ of G. Consider the exact sequence

$$0 \longrightarrow M^G \longrightarrow M \longrightarrow \oplus_{i=1}^n M$$

where the last map is $m \mapsto \oplus_{i=1}^n (g_i - 1)m$. Since B is flat over A, tensoring up with B, we get

$$0 \longrightarrow M^G \otimes_A B \longrightarrow M \otimes_A B \longrightarrow \oplus_{i=1}^m M \otimes_A B.$$

This yields the lemma. $\square$

Convention 2.20. Let X be a rigid analytic space over $\mathbb{Q}_p$. Let G be a group, and let M be a presheaf on X equipped with a G-action. We denote by M^G the presheaf on X defined by $M^G(U) = M(U)^G$ for any admissible open subset U of X.

Proposition 2.21. *The following are true.*

(1) *The presheaf $\mathscr{D}_{dif}^{+,n}(V_S)/(t^k)$ is a locally free coherent sheaf.*

(2) *The presheaf $\mathscr{D}_{dif}^+(V_S)/(t^k)$ is a sheaf.*

(3) *The presheaf $(\mathscr{D}_{dif}^{+,n}(V_S)/(t^k))^\Gamma$ is a coherent sheaf.*

(4) *The presheaf $(\mathscr{D}_{dif}^+(V_S)/(t^k))^\Gamma$ is a sheaf.*

Proof. By the previous lemma, (1) implies (3). By the same argument as in the proof of Proposition 2.9, (1) implies (2) and (3) implies (4) respectively. Thus it suffices to prove (1). Note that $D_{dif}^{+,n}(V_{S'})/(t^k)$ is a locally free S'-module of finite rank for any affinoid subdomain $M(S')$ of $M(S)$. We only need to show that $\mathscr{D}_{dif}^{+,n}(V_S)/(t^k)$ satisfies the sheaf properties. The latter follows from the fact that the presheaf

$$M(S') \mapsto (K_n[[t]] \widehat{\otimes}_{\mathbb{Q}_p} S')/(t^k)$$

is a sheaf as in the proof of Proposition 2.9. $\square$

We also denote the sheaf $\mathscr{D}_{\mathrm{dif}}^+(V_S)/(t)$ by $\mathscr{D}_{\mathrm{Sen}}(V_S)$.

Definition 2.22. Let X be a rigid analytic space over $\mathbb{Q}_p$, and let V_X be a locally free coherent $\mathcal{O}_X$-module equipped with a continuous $\mathcal{O}_X$-linear G_K-action. We choose an admissible covering of X by affinoid subdomains $\{M(S_i)\}_{i \in I}$. We then define the sheaf $\mathscr{D}_{\mathrm{dif}}^+(V_X)$ (resp. $\mathscr{D}_{\mathrm{dif}}^+(V_X)/(t^k)$, $\mathscr{D}_{\mathrm{Sen}}(V_X)$) by gluing the sheaves $\mathscr{D}_{\mathrm{dif}}^+(V_{S_i})$ (resp. $\mathscr{D}_{\mathrm{dif}}^+(V_{S_i})/(t^k)$, $\mathscr{D}_{\mathrm{Sen}}(V_{S_i})$) for all $i \in I$; this construction is independent of the choice of the covering $\{M(S_i)\}_{i \in I}$.

We need the following result in §5.

Proposition 2.23. *If S is torsion-free, then both*

$$(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))^\Gamma \quad and \quad (\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))/(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))^\Gamma$$

are torsion-free S-modules.

Proof. Since $\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k)$ is a finitely generated locally free S-module, it is torsion-free by the assumption on S. So $(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))^\Gamma$ is torsion-free as well. On the other hand, for $s \in S$, $a \in \mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k)$ and $\gamma \in \Gamma$, if $\gamma(sa) = sa$, then $\gamma(a) = a$ because γ acts trivially on S and $\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k)$ is torsion-free. This yields that $(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))/(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))^\Gamma$ is a torison-free S-module. $\square$

2.5. Sen operator. Let V_S be a free S-linear representation of G_K of rank d. Let T_S and L be as in the construction of $\mathrm{D}_K^\dagger(V_S)$, and let $n \geq n(V_S)$. By its construction, the module $\mathrm{D}_{\mathrm{Sen}}^{L_n}(V_S)$ is free of rank d over $L_n \otimes_{\mathbb{Q}_p} S$. Furthermore, for any $\gamma \in \Gamma_L$ satisfying $n(\gamma) \geq n$, we may choose some $L_n \otimes_{\mathbb{Q}_p} S$-basis of $\mathrm{D}_{\mathrm{Sen}}^{L_n}(V_S)$ so that the matrix M_γ of γ under this basis satisfies $|M_\gamma - 1| < 1$. We then define $\log \gamma \in \mathrm{End}_{L_n \otimes_{\mathbb{Q}_p} S}(\mathrm{D}_{\mathrm{Sen}}^{L_n}(V_S))$ by setting

$$\log \gamma = -\sum_{m \geq 1} \frac{(1-\gamma)^m}{m}.$$

The convergence of the right hand side follows from the condition $|M_\gamma - 1| < 1$. Since Γ_L is a 1-dimensional p-adic Lie group, the operator

$$\Theta = \log \gamma / \log_p \chi(\gamma) \in \mathrm{End}_{L_n \otimes_{\mathbb{Q}_p} S}(\mathrm{D}_{\mathrm{Sen}}^{L_n}(V_S))$$

is independent of the choice of γ; hence it is well-defined. Note that

$$\mathrm{D}_{\mathrm{Sen}}^{L_n}(V_S) = \mathrm{D}_{\mathrm{Sen}}^{K_n}(V_S) \otimes_{K_n} L_n,$$

and γ carries $\mathrm{D}_{\mathrm{Sen}}^{K_n}(V_S)$ into itself. Hence we may view Θ as an element of $\mathrm{End}_{K_n \otimes_{\mathbb{Q}_p} S}(\mathrm{D}_{\mathrm{Sen}}^{K_n}(V_S))$. Furthermore, since Γ is commutative, Θ commutes with Γ; hence its characteristic polynomial has coefficients in $(K_n \otimes_{\mathbb{Q}_p} S)^\Gamma = K \otimes_{\mathbb{Q}_p} S$.

Definition 2.24. Let X be a rigid analytic space over $\mathbb{Q}_p$, and let V_X be a locally free coherent $\mathcal{O}_X$-module equipped with a continuous $\mathcal{O}_X$-linear G_K-action. We choose an admissible covering of X by affinoid subdomains $\{M(S_i)\}_{i \in I}$ such that V_{S_i} is free for each $i \in I$. We glue the operators $\Theta \in \mathrm{End}_{K_\infty \otimes_{\mathbb{Q}_p} S_i}(\mathscr{D}_{\mathrm{Sen}}(V_{S_i}))$ for all $i \in I$ to obtain an operator $\Theta \in \mathrm{End}_{K_\infty \otimes_{\mathbb{Q}_p} \mathcal{O}_X}(\mathscr{D}_{\mathrm{Sen}}(V_X))$; this is independent of the choice of the covering. We call Θ the *Sen operator* for V_X.

We also glue the characteristic polynomials of $\Theta \in \mathrm{End}_{K_\infty \otimes_{\mathbb{Q}_p} S_i}(\mathscr{D}_{\mathrm{Sen}}(V_{S_i}))$ for all $i \in I$ to get an element of $(K \otimes_{\mathbb{Q}_p} \mathcal{O}(X))[T]$; this is independent of the choice of the covering, and it is called the *Sen polynomial* for V_X.

Remark 2.25. By their constructions and base change property of $\mathrm{D}_{\mathrm{Sen}}$, it is clear that the notions of Sen operator and Sen polynomial compatible with base change. That is, given a morphism $f : X' \to X$ of $\mathbb{Q}_p$-rigid analytic spaces, the Sen operator and Sen polynomial of $f^* V_X$ are naturally isomorphic to the pullback of the Sen operator and Sen polynomial of V_X via f respectively.

The rest of this subsection is a (φ, Γ)-module theoretical interpretation of [25, (2.3)–(2.6)].

Proposition 2.26. *Let V_S be a finite free S-linear representation. Then for any $n \geq n(V_S)$, both $H^0(\Gamma, \mathrm{D}_{\mathrm{Sen}}^n(V_S))$ and $H^1(\Gamma, \mathrm{D}_{\mathrm{Sen}}^n(V_S))$ are killed by $\det(\Theta)$.*

Proof. Let L, γ be as above. Since $H^0(\Gamma_L, \mathrm{D}_{\mathrm{Sen}}^n(V_S))$ and $H^1(\Gamma_L, \mathrm{D}_{\mathrm{Sen}}^n(V_S))$ are computed by the complex

$$0 \to \mathrm{D}_{\mathrm{Sen}}^n(V_S) \xrightarrow{\gamma - 1} \mathrm{D}_{\mathrm{Sen}}^n(V_S) \to 0,$$

both of them are killed by $\gamma - 1$. Thus both of them are killed by Θ; hence both of them are killed by $\det(\Theta)$. This yields the desired result since $H^0(\Gamma, \mathrm{D}_{\mathrm{Sen}}^n(V_S)) \subseteq H^0(\Gamma_L, \mathrm{D}_{\mathrm{Sen}}^n(V_S))$ and $H^1(\Gamma, \mathrm{D}_{\mathrm{Sen}}^n(V_S))$ is a quotient of $H^1(\Gamma_L, \mathrm{D}_{\mathrm{Sen}}^n(V_S))$. $\square$

From now on, let V_S be only locally free over S.

Corollary 2.27. *For any $k \geq 1$ and $n \geq n(V_S)$, the natural map*

$$(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))^\Gamma \to (\mathrm{D}_{\mathrm{Sen}}^n(V_S))^\Gamma$$

has kernel and cokernel killed by $\prod_{i=1}^{k-1} \det(\Theta + iI)$.

Proof. Since $(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))^\Gamma$ and $(\mathscr{D}_{\mathrm{Sen}}^n(V_S))^\Gamma$ are coherent sheaves, by restricting on a finite covering of $M(S)$, it suffices to treat the case that V_S is free over S. It then suffices to show that the natural map

$$(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^{i+1}))^\Gamma \to (\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^i))^\Gamma$$

has kernel and cokernel killed by $\det(\Theta + iI)$ for each $i \geq 1$. By the short exact sequence

$$0 \longrightarrow \mathrm{D}_{\mathrm{Sen}}^n(V_S(i)) \longrightarrow \mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^{i+1}) \longrightarrow \mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^i) \longrightarrow 0,$$

we get the exact sequence

$$0 \to (\mathrm{D}^n_{\mathrm{Sen}}(V_S(i)))^\Gamma \to (\mathrm{D}^{+,n}_{\mathrm{dif}}(V_S)/(t^{i+1}))^\Gamma$$
$$\to (\mathrm{D}^{+,n}_{\mathrm{dif}}(V_S)/(t^i))^\Gamma \to H^1(\Gamma, \mathrm{D}^n_{\mathrm{Sen}}(V_S(i))).$$

We thus conclude from Proposition 2.26 and the fact that Sen operator for $V_S(i)$ is $\Theta + iI$. □

Proposition 2.28. *Keep notations as above. Then there exists a finite Galois extension L' of K containing L such that $\Theta/(\gamma' - 1)$ is invertible on $\mathrm{D}^n_{\mathrm{Sen}}(V_S)$ for any $\gamma' \in \Gamma_{L'}$.*

Proof. It suffices to treat the case that V_S is free. Let $\gamma \in \Gamma_L$ such that $\chi(\gamma) \in 1 + p^n \mathbb{Z}_p$. It follows that γ acts $L_n \otimes_{\mathbb{Q}_p} S$-linearly on $\mathrm{D}^{L_n}_{\mathrm{Sen}}(V_S)$. Thus for any positive integer k, the matrix $M_{\gamma^{p^k}}$ of γ^{p^k} is just $M_\gamma^{p^k}$. Therefore we may choose a sufficiently large k so that $|M_{\gamma^{p^k}} - 1| < p^{-1}$. Let L' be a finite Galois extension of K so that $\Gamma_{L'} \subseteq \langle \gamma^{p^k} \rangle$. Then for any $\gamma' \in \Gamma_{L'}$, we have $|M_{\gamma'} - 1| \leq |M_{\gamma^{p^k}} - 1| < p^{-1}$. It follows that

$$|(M_{\gamma'} - 1)^m/(m+1)| < \frac{p^{-m}}{|m+1|} \leq p^{-1}$$

for any $m \geq 1$. Let $u = \sum_{m \geq 0}(1 - \gamma')^m/(m+1)$. It follows that the matrix of $u - 1$, which is

$$\sum_{m \geq 1}(1 - M_{\gamma'})^m/(m+1),$$

has positive valuation. This yields that u is invertible. Hence $\Theta/(\gamma' - 1) = \chi(\gamma')^{-1}u$ is invertible. □

In the following, we further suppose $\det(\Theta) = 0$, and write $\det(TI - \Theta) = TQ(T)$ for some $Q(T) \in (K \otimes_{\mathbb{Q}_p} S)[T]$. Put $P(i) = \prod_{j=0}^{i-1} Q(-j)$ for every integer $i \geq 1$.

Proposition 2.29. *If $f : S \to R$ is a map of affinoid algebras over $\mathbb{Q}_p$, for each $n \geq n(V_S)$, the natural map*

$$(\mathrm{D}^n_{\mathrm{Sen}}(V_S))^\Gamma \otimes_S R \to (\mathrm{D}^n_{\mathrm{Sen}}(V_R))^\Gamma \tag{2.6}$$

has kernel and cokernel killed by a power of $f(Q(0))$. In particular, if $f(Q(0))$ is a unit, this map is an isomorphism.

Proof. Write $Q(T) = \sum_{i=0}^{d-1} a_i T^i$. First note that $Q(\Theta)\Theta = 0$ in $\mathrm{End}(\mathrm{D}_{\mathrm{Sen}}(V_S))$ by Cayley's theorem. Hence

$$\Theta(\mathrm{D}^n_{\mathrm{Sen}}(V_S)) \subseteq \ker(Q(\Theta)|\mathrm{D}^n_{\mathrm{Sen}}(V_S)) \quad \text{and} \quad Q(\Theta)(\mathrm{D}^n_{\mathrm{Sen}}(V_S)) \subseteq \ker(\Theta|\mathrm{D}^n_{\mathrm{Sen}}(V_S)).$$

By the equality $a_0 = Q(\Theta) - \Theta(\sum_{i=1}^{d-1} a_i \Theta^{i-1})$, we deduce that both the kernel and cokernel of the natural map

$$\ker(\Theta|\mathrm{D}_{\mathrm{Sen}}^n(V_S)) \oplus \ker(Q(\Theta)|\mathrm{D}_{\mathrm{Sen}}^n(V_S)) \longrightarrow \mathrm{D}_{\mathrm{Sen}}^n(V_S)$$

are killed by $a_0 = Q(0)$. Hence the natural map

$$(\ker(\Theta|\mathrm{D}_{\mathrm{Sen}}^n(V_S)))_{a_0} \oplus (\ker(Q(\Theta)|\mathrm{D}_{\mathrm{Sen}}^n(V_S)))_{a_0} \longrightarrow (\mathrm{D}_{\mathrm{Sen}}^n(V_S))_{a_0} \qquad (2.7)$$

is an isomorphism. By the same reasoning, the natural map

$$(\ker(\Theta|\mathrm{D}_{\mathrm{Sen}}^n(V_R)))_{f(a_0)} \oplus (\ker(Q(\Theta)|\mathrm{D}_{\mathrm{Sen}}^n(V_R)))_{f(a_0)} \longrightarrow (\mathrm{D}_{\mathrm{Sen}}^n(V_R))_{f(a_0)}$$

is also an isomorphism. Consider the following commutative diagram

$$\ker(\Theta|\mathrm{D}_{\mathrm{Sen}}^n(V_S)) \otimes_S R_{f(a_0)} \oplus \ker(Q(\Theta)|\mathrm{D}_{\mathrm{Sen}}^n(V_S)) \otimes_S R_{f(a_0)} \longrightarrow \mathrm{D}_{\mathrm{Sen}}^n(V_S) \otimes_S R_{f(a_0)}$$

$$\downarrow \hspace{8cm} \downarrow$$

$$(\ker(\Theta|\mathrm{D}_{\mathrm{Sen}}^n(V_R)))_{f(a_0)} \oplus (\ker(Q(\Theta)|\mathrm{D}_{\mathrm{Sen}}^n(V_R)))_{f(a_0)} \longrightarrow (\mathrm{D}_{\mathrm{Sen}}^n(V_R))_{f(a_0)}$$

where the upper map, which is obtained by tensoring up (2.7) with R over S, is an isomorphism. Note that the right map is an isomorphism because $\mathrm{D}_{\mathrm{Sen}}^n(\cdot)$ is functorial in V_S. We thus deduce that both the natural maps

$$\ker(\Theta|\mathrm{D}_{\mathrm{Sen}}^n(V_S)) \otimes_S R_{f(a_0)} \to (\ker(\Theta|\mathrm{D}_{\mathrm{Sen}}^n(V_R)))_{f(a_0)}$$

and

$$\ker(Q(\Theta)|\mathrm{D}_{\mathrm{Sen}}^n(V_S)) \otimes_S R_{f(a_0)} \to (\ker(Q(\Theta)|\mathrm{D}_{\mathrm{Sen}}^n(V_R)))_{f(a_0)}$$

are isomorphisms. Let L' be a finite Galois extension of K given by Proposition 2.28. It then follows from Proposition 2.28 that

$$\ker(\Theta|\mathrm{D}_{\mathrm{Sen}}^n(V_S)) = (\mathrm{D}_{\mathrm{Sen}}^n(V_S))^{\Gamma_{L_n'}}$$

and

$$\ker(\Theta|\mathrm{D}_{\mathrm{Sen}}^n(V_R)) = (\mathrm{D}_{\mathrm{Sen}}^n(V_R))^{\Gamma_{L_n'}}.$$

Note that $(\mathrm{D}_{\mathrm{Sen}}^n(V_S))^\Gamma$ (resp. $(\mathrm{D}_{\mathrm{Sen}}^n(V_R))^\Gamma$) is the image of the endomorphism

$$a \mapsto \sum_{g \in \Gamma/\Gamma_{L_n'}} ga$$

on $(\mathrm{D}_{\mathrm{Sen}}^n(V_S))^{\Gamma_{L_n'}}$ (resp. $(\mathrm{D}_{\mathrm{Sen}}^n(V_R))^{\Gamma_{L_n'}}$). We therefore conclude immediately that the natural map

$$(\mathrm{D}_{\mathrm{Sen}}^n(V_S))^{\Gamma_K} \otimes_S R_{f(a_0)} \longrightarrow (\mathrm{D}_{\mathrm{Sen}}^n(V_R))^{\Gamma_K}_{f(a_0)}$$

is an isomorphism. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2.30. *If* $f : S \to R$ *is a map of affinoid algebras over* $\mathbb{Q}_p$, *for each* $n \geq n(V_S)$, *the natural map*

$$(D_{\mathrm{dif}}^{+,n}(V_S)/(t^k))^{\Gamma} \otimes_S R \to (D_{\mathrm{dif}}^{+,n}(V_R)/(t^k))^{\Gamma}$$

has kernel and cokernel killed by a power of $f(P(k))$. *In particular, if* $f(P(k))$ *is a unit, this map is an isomorphism.*

Proof. Consider the following commutative diagram

$$
\begin{array}{ccc}
(D_{\mathrm{dif}}^{+,n}(V_S)/(t^k))^{\Gamma} \otimes_S R_{f(P(k))} & \longrightarrow & (D_{\mathrm{dif}}^{+,n}(V_R)/(t^k))^{\Gamma}_{f(P(k))} \\
\downarrow & & \downarrow \\
(D_{\mathrm{Sen}}^{n}(V_S))^{\Gamma} \otimes_S R_{f(P(k))} & \longrightarrow & (D_{\mathrm{Sen}}^{n}(V_R))^{\Gamma}_{f(P(k))}.
\end{array}
$$

The bottom map is an isomorphism by Proposition 2.29. The left map and right map are isomorphisms by Corollary 2.27. Hence the upper map is an isomorphism; this yields the desired result. $\qquad\square$

3. The extended Robba ring

3.1. Definitions. Let B be a $\mathbb{Q}_p$-Banach algebra with $|B|$ discrete. Set $v(x) = -\log_p(|x|)$ for any $x \in B$.

Definition 3.1. For any interval $I \subseteq (0, \infty)$, let $\mathcal{R}_B^I$ be the ring of Laurent series

$$f = \sum_{i \in \mathbb{Z}} a_i T^i$$

for which $a_i \in B$ and $v(a_i) + si \to \infty$ as $i \to \pm\infty$ for all $s \in I$. For any $s \in I$, define $w_s : \mathcal{R}_B^I \to \mathbb{R}$ as

$$w_s(f) = \min_{i \in \mathbb{Z}}\{v(a_i) + si\}$$

and the norm $|\cdot|_s$ on $\mathcal{R}_B^I$ as

$$|f|_s = \max_{i \in \mathbb{Z}}\{|a_i|p^{-si}\} = p^{-w_s(f)}.$$

We denote $\mathcal{R}_B^{(0,r]}$ by $\mathcal{R}_B^r$ for simplicity. Let $\mathcal{R}_B^{\mathrm{bd},r}$ be the subring of $\mathcal{R}_B^r$ consisting of elements with $\{v(a_i)\}_{i \in \mathbb{Z}}$ bounded below. Define $w : \mathcal{R}_B^{\mathrm{bd},r} \to \mathbb{R}$ as

$$w(f) = \min_{i \in \mathbb{Z}}\{v(a_i)\}.$$

Let $\mathcal{R}_B^{\mathrm{int},r}$ be the subring of $\mathcal{R}_S^{\mathrm{bd},r}$ consisting of f with $w(f) \geq 0$. We call $\mathcal{R}_B = \cup_{r>0}\mathcal{R}_B^r$ the *Robba ring over* B, and call $\mathcal{R}_B^{\mathrm{bd}} = \cup_{r>0}\mathcal{R}_B^r$ the *bounded Robba ring over* B.

Definition 3.2. For any interval $I \subseteq (0, \infty)$, let $\widetilde{\mathcal{R}}_B^I$ be the set of formal sums

$$f = \sum_{i \in \mathbb{Q}} a_i u^i$$

with $a_i \in B$ satisfying the following conditions.

(1) For any $c > 0$, the set of $i \in \mathbb{Q}$ so that $|a_i| \geq c$ is well-ordered (i.e. has no infinite decreasing subsequence).

(2) For all $s \in I$, $v(a_i) + si \to \infty$ as $i \to \pm\infty$, and $\inf_{i \in \mathbb{Q}}\{v(a_i) + si\} > -\infty$.

These series form a ring under formal series addition and multiplication. For any $s \in I$, set $w_s : \widetilde{\mathcal{R}}_B^I \to \mathbb{R}$ as

$$w_s(f) = \inf_{i \in \mathbb{Q}}\{v(a_i) + si\}$$

and the norm $|f|_s$ on $\widetilde{\mathcal{R}}_B^I$ as

$$|f|_s = \sup_{i \in \mathbb{Q}}\{|a_i|p^{-si}\} = p^{-w_s(f)}.$$

We denote $\widetilde{\mathcal{R}}_B^{(0,r]}$ by $\widetilde{\mathcal{R}}_B^r$ for simplicity. Let $\widetilde{\mathcal{R}}_B^{\mathrm{bd},r}$ be the subring of $\widetilde{\mathcal{R}}_B^r$ consisting of elements f with $\{v(a_i)\}_{i \in \mathbb{Q}}$ bounded below. Define $w : \widetilde{\mathcal{R}}_B^{\mathrm{bd},r} \to \mathbb{R}$ as

$$w(f) = \min_{i \in \mathbb{Q}}\{v(a_i)\}.$$

We call $\widetilde{\mathcal{R}}_B = \cup_{r>0}\widetilde{\mathcal{R}}_B^r$ the *extended Robba ring over B*, and call $\widetilde{\mathcal{R}}_B^{\mathrm{bd}} = \cup_{r>0}\widetilde{\mathcal{R}}_B^r$ the *extended bounded Robba ring over B*.

We refer the reader to §3.3 for more discussion of the extended Robba ring.

Remark 3.3. Since $|B|$ is discrete, it follows from condition (1) that $\inf_{i \in \mathbb{Q}}\{v(a_i) + si\}$ (hence also $\sup_{i \in \mathbb{Q}}\{|a_i|p^{-si}\}$) is attained at some $i \in \mathbb{Q}$.

We equip $\mathcal{R}_B^I$ (resp. $\widetilde{\mathcal{R}}_B^I$) with the Fréchet topology defined by $\{w_s\}_{s \in I}$; then $\mathcal{R}_B^I$ (resp. $\widetilde{\mathcal{R}}_B^I$) is a complete Fréchet algebra over $\mathbb{Q}_p$. Furthermore, in the case that $I = [a, b]$ is a closed interval, $\mathcal{R}_B^I$ (resp. $\widetilde{\mathcal{R}}_B^I$) is a Banach algebra over $\mathbb{Q}_p$ with the norm $\max\{w_a, w_b\}$. We equip $\mathcal{R}_B^{\mathrm{bd},r}$ (resp. $\widetilde{\mathcal{R}}_B^{\mathrm{bd},r}$) with the norm $\max\{w, w_r\}$; then $\mathcal{R}_B^{\mathrm{bd},r}$ (resp. $\widetilde{\mathcal{R}}_B^{\mathrm{bd},r}$) is a Banach algebra over $\mathbb{Q}_p$.

Definition 3.4. Let $\widetilde{\mathcal{E}}_B$ be the ring of formal sums $f = \sum_{i \in \mathbb{Q}} a_i u^i$ with $a_i \in B$ satisfying the following conditions.

(1) For each $c > 0$, the set of $i \in \mathbb{Q}$ such that $|a_i| \geq c$ is well-ordered.

(2) The set $\{v(a_i)\}_{i \in \mathbb{Q}}$ is bounded below and $v(a_i) \to \infty$ as $i \to -\infty$.

Set $w : \widetilde{\mathcal{E}}_B \to \mathbb{R}$ as

$$w(f) = \min_{i \in \mathbb{Q}}\{v(a_i)\}.$$

We equip $\widetilde{\mathcal{E}}_B$ with the topology defined by w; then $\widetilde{\mathcal{E}}_B$ is complete for this topoology.

In the following, let L be a p-adic field in the sense that it is a complete discretely valued field equipped with the structure of an $\mathbb{Q}_p$-algebra in such a way that the map $\mathbb{Q}_p \to L$ is continuous. Put $B_L = L \widehat{\otimes}_{\mathbb{Q}_p} B$.

Proposition 3.5. *For* $R \in \{\mathcal{R}^{\mathrm{bd},r}, \mathcal{R}^I\}$ *or* $R = \mathcal{R}^r$, *where* $I \subset (0, \infty)$ *is a closed interval, the natural map* $R_L \otimes_{\mathbb{Q}_p} B \to R_{B_L}$ *induces an isometric isomorphism*

$$R_L \widehat{\otimes}_{\mathbb{Q}_p} B \cong R_{B_L}$$

of L-Banach algebras or Fréchet algebras. For $\widetilde{R} \in \{\widetilde{\mathcal{E}}, \widetilde{\mathcal{R}}^{\mathrm{bd},r}, \widetilde{\mathcal{R}}^I\}$ *or* $\widetilde{R} = \widetilde{\mathcal{R}}^r$, *the natural map* $\widetilde{R}_L \otimes_{\mathbb{Q}_p} B \to \widetilde{R}_{B_L}$ *induces an isometric embedding*

$$\widetilde{R}_L \widehat{\otimes}_{\mathbb{Q}_p} B \hookrightarrow \widetilde{R}_{B_L}$$

of L-Banach algebras or Fréchet algebras.

Proof. This follows from [27, Lemma 2.1.6]. □

Proposition 3.6. *If B is of countable type, then*

$$(\widetilde{\mathcal{E}}_L \widehat{\otimes}_{\mathbb{Q}_p} B) \cap \widetilde{\mathcal{R}}^{\mathrm{bd},r}_{B_L} = \widetilde{\mathcal{R}}^{\mathrm{bd},r}_L \widehat{\otimes}_{\mathbb{Q}_p} B.$$

Proof. This follows from [27, Lemma 2.1.8] by taking $S = \widetilde{\mathcal{E}}_L$. □

Lemma 3.7. *Let S be an affinoid algebra over $\mathbb{Q}_p$. Then for any $x \in M(S)$, the natural map $\mathcal{R}^r_S \otimes_S k(x) \to \mathcal{R}^r_{k(x)}$ is an isomorphism.*

Proof. It reduces to show that the natural map $\rho_x : \mathcal{R}^r_S \to \mathcal{R}^r_{k(x)}$ is surjective and its kernel is $\mathfrak{m}_x \mathcal{R}^r_S$. By [6, Proposition A.2.2], the exact sequence

$$0 \longrightarrow \mathfrak{m}_x \longrightarrow S \longrightarrow k(x) \longrightarrow 0$$

induces the exact sequence

$$0 \longrightarrow \mathcal{R}^r_{\mathbb{Q}_p} \widehat{\otimes}_{\mathbb{Q}_p} \mathfrak{m}_x \longrightarrow \mathcal{R}^r_{\mathbb{Q}_p} \widehat{\otimes}_{\mathbb{Q}_p} S \longrightarrow \mathcal{R}^r_{\mathbb{Q}_p} \widehat{\otimes}_{\mathbb{Q}_p} k(x) \longrightarrow 0.$$

Using Proposition 3.5, we get that ρ_x is surjective. Choose a finite set of generators $b_1, \ldots, b_m$ of $\mathfrak{m}_x$. By the open mapping theorem for Banach spaces over discretely valued fields, the surjective map of $\mathbb{Q}_p$-Banach spaces $S^m \to \mathfrak{m}_x$ defined by $(a_1, \ldots, a_m) \mapsto \sum_{i=1}^m a_i b_i$ is open. Hence there exists $c > 0$ such that for any $a \in \mathfrak{m}_x$, there exist $a_1, \ldots, a_m \in S$ with $|a_i| \leq c|a|$ such that $a = \sum_{i=1}^m a_i b_i$. Now let $f = \sum_{i \in \mathbb{Q}} a_i u^i$ belongs to kernel of ρ_x; so $a_i \in \mathfrak{m}_x$ for all i. For each $i \in \mathbb{Q}$, choose $a_{ij} \in S$ with $|a_{ij}| \leq c|a_i|$ for $1 \leq j \leq m$ such that $a_i = \sum_{j=1}^m a_{ij} b_j$. Let $f_j = \sum_{i \in \mathbb{Q}} a_{ij} u^i$ for $1 \leq j \leq m$. It is then clear that $f_j \in \mathcal{R}^r_S$ and $f = \sum_{j=1}^m b_j f_j$; hence $f \in \mathfrak{m}_x \mathcal{R}^r_S$. □

3.2. Key lemma. From now on, suppose that L is equipped with an isometric automorphism φ_L such that its restriction on $\mathbb{Q}_p$ is the identity. Let S be an affinoid algebra over $\mathbb{Q}_p$, and let φ be the continuous extension of $\varphi_L \otimes \mathrm{id}$ to S_L. We fix a positive integer $q > 1$, and we extend φ to automorphisms on $\widetilde{\mathcal{R}}_{S_L}$ and $\widetilde{\mathcal{E}}_{S_L}$ by setting

$$\varphi\left(\sum_{i \in \mathbb{Q}} a_i u^i\right) = \sum_{i \in \mathbb{Q}} \varphi(a_i) u^{qi}.$$

It is obvious that φ restricts to automorphisms on $\widetilde{\mathcal{R}}_L \widehat{\otimes}_{\mathbb{Q}_p} S$ and $\widetilde{\mathcal{E}}_L \widehat{\otimes}_{\mathbb{Q}_p} S$.

Let $\alpha \in S^\times$. Consider the following Frobenius equation

$$\varphi(b) - \alpha b = a. \tag{3.1}$$

The following is a variant of [27, Lemma 2.3.5(3)].

Lemma 3.8. *Suppose $|\alpha^{-1}| < 1$. Then for $a = \sum_{i \in \mathbb{Q}} a_i u^i \in \widetilde{\mathcal{R}}^r_{S_L}$, the following are true.*

(1) *(3.1) admits at most one solution $b \in \widetilde{\mathcal{R}}_{S_L}$.*

(2) *(3.1) has a solution $b \in \widetilde{\mathcal{R}}_{S_L}$ if and only if*

$$\sum_{m \in \mathbb{Z}} \alpha^{-(m+1)} \varphi^m(a_{iq^{-m}}) = 0 \tag{3.2}$$

for all $i < 0$. Furthermore, in this case the unique solution b is given by

$$b = -\sum_{i \in \mathbb{Q}} \left(\sum_{m \in \mathbb{N}} \alpha^{-(m+1)} \varphi^m(a_{iq^{-m}})\right) u^i$$

and belongs to $\widetilde{\mathcal{R}}^{qr}_{S_L}$, and it satisfies $w_r(b) \geq w_r(a) - C(r, \alpha)$ where $C(r, \alpha)$ is some constant only depending on r, α.

(3) *Suppose $a \in \widetilde{\mathcal{R}}^r_L \widehat{\otimes}_{\mathbb{Q}_p} S$. If $b \in \widetilde{\mathcal{R}}_{S_L}$ is a solution of (3.8), then $b \in \widetilde{\mathcal{R}}^r_L \widehat{\otimes}_{\mathbb{Q}_p} S$.*

Proof. Suppose that $b = \sum_{i \in \mathbb{Q}} b_i u^i \in \widetilde{\mathcal{R}}^{r'}_{S_L}$ is a solution of (3.1). By comparing coefficients, we get

$$\varphi(b_{i/q}) - \alpha b_i = a_i,$$

yielding

$$b_i = \alpha^{-1} \varphi(b_{i/q}) - \alpha^{-1} a_i \tag{3.3}$$

for every $i \in \mathbb{Q}$. Since $|\alpha^{-1}|_{\mathrm{sp}} < 1$ and $\{|a_{iq^{-m}}|\}_{m \in \mathbb{N}}$ are bounded, we get

$$b_i = -\sum_{m \in \mathbb{N}} \alpha^{-(m+1)} \varphi^m(a_{iq^{-m}})$$

by iterating (3.3). Thus b_i is uniquely determined by α and a. This proves (1).

Furthermore, for any $k \in \mathbb{N}$,

$$\varphi^k \left(\sum_{m=-k}^{\infty} \alpha^{-(m+1)} \varphi^m (a_{iq^{-m}}) \right) = \alpha^k \sum_{m=0}^{\infty} \alpha^{-(m+1)} \varphi^m (a_{(iq^k)p^{-m}}) = -\alpha^k b_{iq^k}.$$

Hence

$$v \left(\sum_{m=-k}^{\infty} \alpha^{-(m+1)} \varphi^m (a_{iq^{-m}}) \right) = v(\alpha^k b_{iq^k}) \geq k v(\alpha) + v(b_{iq^k})$$

$$\geq k v(\alpha) + w_{r'}(b) - r' i q^k.$$

It follows that if $i < 0$, then $v(\sum_{m=-k}^{\infty} \alpha^{-(m+1)} \varphi^m (a_{iq^{-m}})) \to \infty$ as $k \to \infty$; this yields (3.2), proving the "only if" part of (2).

To prove the "if" part of (2), for $f = \sum_{i \in \mathbb{Q}} a_i u^i \in \widetilde{\mathcal{R}}_{S_L}^r$ and $c \in \mathbb{R}$, we set

$$w_r^{c,-}(f) = \min_{i \leq c} \{ v(a_i) + ri \}.$$

It is clear that $w_r^{c,-}(f) \to \infty$ as $c \to -\infty$. Now suppose that (3.2) holds for all $i < 0$. If $i \leq -1$, then for each $m \leq -1$,

$$v(\alpha^{-(m+1)} \varphi^m (a_{iq^{-m}})) \geq v(a_{iq^{-m}}) - (m+1)v(\alpha)$$

$$= (v(a_{iq^{-m}}) + riq^{-m}) - riq^{-m} - (m+1)v(\alpha)$$

$$\geq (w_r^{i,-}(a) - ri) + ri(1 - q^{-m}) - (m+1)v(\alpha)$$

$$\geq (w_r^{i,-}(a) - ri) + r(q^{-m} - 1) - (m+1)v(\alpha)$$

$$\geq w_r^{i,-}(a) - ri - C_1(r, \alpha),$$

where $C_1(r, \alpha)$ is some constant depending on r, α. Hence

$$w_r \left(\left(\sum_{m=0}^{\infty} \alpha^{-(m+1)} \varphi^m (a_{iq^{-m}}) \right) u^i \right) = v \left(- \sum_{m=-1}^{-\infty} \alpha^{-(m+1)} \varphi^m (a_{iq^{-m}}) \right) + ri$$

$$\geq w_r^{i,-}(a) - C_1(r, \alpha)$$

$$(3.4)$$

for each $i \leq -1$. Note that $v(\alpha^{-1}) > 0$ by assumption. Thus if $-1 < i < 0$, for any $m \geq 0$,

$$v(\alpha^{-(m+1)} \varphi^m (a_{iq^{-m}})) \geq w_r(a) - riq^{-m} + v(\alpha^{-1})(m+1)$$

$$> w_r(a) - riq^{-m} \qquad (3.5)$$

$$> w_r(a) - ri.$$

Hence

$$w_r \left(\left(\sum_{m=0}^{\infty} \alpha^{-(m+1)} \varphi^m (a_{iq^{-m}}) \right) u^i \right) \geq w_r(a) \qquad (3.6)$$

for all $i > -1$.

Now put

$$a^+ = \sum_{i \geq 0} a_i u^i, \qquad a^- = \sum_{i < 0} a_i u^i$$

and

$$b^+ = -\sum_{i \geq 0} \left(\sum_{m=0}^{\infty} \alpha^{-(m+1)} \varphi^m(a_{iq^{-m}}) \right) u^i,$$

$$b^- = -\sum_{i < 0} \left(\sum_{m=0}^{\infty} \alpha^{-(m+1)} \varphi^m(a_{iq^{-m}}) \right) u^i.$$

Since $|\alpha^{-1}| < 1$, it is straightforward to see that the series $\sum_{m=0}^{\infty} \alpha^{-(m+1)} \varphi^m(a^+)$ is convergent in $\widetilde{\mathcal{R}}^r_{S_L}$, and

$$w_r \left(\sum_{m=0}^{\infty} \alpha^{-(m+1)} \varphi^m(a^+) \right) \geq w_r(a^+) \geq w_r(a).$$

We then deduce

$$\sum_{m=0}^{\infty} \alpha^{-(m+1)} \varphi^m(a^+) = -b^+$$

by comparing the coefficients. We claim that b^- also belongs to $\widetilde{\mathcal{R}}^r_{S_L}$. We first deduce from (3.4) and (3.6) that b^- satisfies Definition 3.2(2). On the other hand, since $|\alpha^{-1}| < 1$, the series $\sum_{m=0}^{\infty} \alpha^{-(m+1)} \varphi^m(a^-)$ is convergent to $-b^-$ in $\widetilde{\mathcal{E}}_{S_L}$. Hence b^- also satisfies Definition 3.2(1), yielding the claim.

Now put $b = b^+ + b^- \in \widetilde{\mathcal{R}}^r_{S_L}$. It is then clear that b is the solution of (3.1). By (3.4) and (3.6), we get

$$w_r(b^-) \geq w_r(a) - C(r, \alpha).$$

where $C(r, \alpha) = \max\{0, C_1(r, \alpha)\}$. It then follows $w_r(b) \geq w_r(a) - C(r, \alpha)$. Furthermore, since $\varphi(b) = a - \alpha b \in \widetilde{\mathcal{R}}^r_{S_L}$, we get $b \in \widetilde{\mathcal{R}}^{qr}_{S_L}$.

It remains to prove (3). If $a \in \widetilde{\mathcal{R}}^r_L \widehat{\otimes}_{\mathbb{Q}_p} S$, then $a^+ \in \widetilde{\mathcal{R}}^r_L \widehat{\otimes}_{\mathbb{Q}_p} S$ and $a^- \in \widetilde{\mathcal{R}}^{\mathrm{bd},r}_L \widehat{\otimes}_{\mathbb{Q}_p} S$. We thus deduce $b^+ \in \widetilde{\mathcal{R}}^r_L \widehat{\otimes}_{\mathbb{Q}_p} S$ and $b^- \in \widetilde{\mathcal{E}}_L \widehat{\otimes}_{\mathbb{Q}_p} S$. Since $b^- \in \widetilde{\mathcal{R}}^{\mathrm{bd},r}_L \widehat{\otimes}_{\mathbb{Q}_p} S$, by Proposition 3.6, we conclude that

$$b^- \in (\widetilde{\mathcal{E}}_L \widehat{\otimes}_{\mathbb{Q}_p} S) \cap \widetilde{\mathcal{R}}^{\mathrm{bd},r}_{S_L} = \widetilde{\mathcal{R}}^{\mathrm{bd},r}_L \widehat{\otimes}_{\mathbb{Q}_p} S.$$

Hence $b = b^+ + b^- \in \widetilde{\mathcal{R}}^r_L \widehat{\otimes}_{\mathbb{Q}_p} S$.   □

Remark 3.9. One can reformulate the above lemma using the notion of cohomology of φ-modules. For any $\alpha \in S^\times$, we define the rank 1 φ-module $\widetilde{\mathcal{R}}_{S_L}(\alpha)$ over $\widetilde{\mathcal{R}}_{S_L}$ by setting $\varphi(v) = \alpha^{-1} v$ for a generator v; we set $H^1(\widetilde{\mathcal{R}}_{S_L}(\alpha)) = \widetilde{\mathcal{R}}_{S_L}(\alpha)/(\varphi - 1)$. Then Lemma 3.8 says that if $|\alpha^{-1}| < 1$, then av is a coboundary if and only if a satisfies (3.2).

3.3. Relations between different rings. Recall that there exists a natural identification $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \cong \Gamma^{\mathrm{alg}}_{\mathrm{an}}$ which identifies $\widetilde{\mathbf{B}}^{\dagger,\rho(r)}_{\mathrm{rig}}$ with $\Gamma^{\mathrm{alg}}_{\mathrm{an},r}$ for any $r > 0$ (see for instance [3, §1.1]); here $\Gamma^{\mathrm{alg}}_{\mathrm{an}}$ and $\Gamma^{\mathrm{alg}}_{\mathrm{an},r}$ are relative extended Robba rings[7] associated to the residue field $\widehat{\mathbb{F}_p((u))^{\mathrm{alg}}}$[8] introduced by Kedlaya (see [20, §2] for more details). On the other hand, $\widetilde{\mathcal{R}}_{\widehat{\mathbb{Q}^{\mathrm{ur}}_p}}$ and $\widetilde{\mathcal{R}}^r_{\widehat{\mathbb{Q}^{\mathrm{ur}}_p}}$ (together with the φ-action for $q = p$) are relative extended Robba rings with residue field $\mathbb{F}^{\mathrm{alg}}_p((u^{\mathbb{Q}}))$; here $\mathbb{F}^{\mathrm{alg}}_p((u^{\mathbb{Q}}))$ is the Hahn–Mal'cev–Neumann algebra with coefficients in $\mathbb{F}^{\mathrm{alg}}_p$ (see for instance [20, Definition 4.5.4]). By [19, Theorem 8], $\widehat{\mathbb{F}_p((u))^{\mathrm{alg}}}$ is a closed subfield of $\mathbb{F}^{\mathrm{alg}}_p((u^{\mathbb{Q}}))$. This leads to natural closed embeddings of $\mathbb{Q}_p$-Fréchet algebras

$$\Gamma^{\mathrm{alg}}_{\mathrm{an},r} \hookrightarrow \widetilde{\mathcal{R}}^r_{\widehat{\mathbb{Q}^{\mathrm{ur}}_p}}$$

for all $r > 0$. By the above identifications, we therefore get closed embeddings

$$\widetilde{\mathbf{B}}^{\dagger,\rho(r)}_{\mathrm{rig}} \hookrightarrow \widetilde{\mathcal{R}}^r_{\widehat{\mathbb{Q}^{\mathrm{ur}}_p}} \qquad (3.7)$$

which respect the φ-action. Henceforth we regard $\widetilde{\mathbf{B}}^{\dagger,\rho(r)}_{\mathrm{rig}}$ as a subring of $\widetilde{\mathcal{R}}^r_{\widehat{\mathbb{Q}^{\mathrm{ur}}_p}}$; we therefore regard $K \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger,\rho(r)}_{\mathrm{rig}}$ as a subring of $\widetilde{\mathcal{R}}^r_{\widehat{K^{\mathrm{ur}}}}$.

We will need the following results in §4.

Lemma 3.10. *For any $a \in S_L$, there exists an analytic subspace $M(S(a))$ of $M(S)$ such that for any map $g : S \to R$ of affinoid algebras over $\mathbb{Q}_p$, $g_L(a) = 0$ if and only if the map $M(R) \to M(S)$ factors through $M(S(a))$.*

Proof. Choose an orthonormal basis $\{e_j\}_{j \in J}$ of L over $\mathbb{Q}_p$; then it is also an orthonormal basis of S_L as an S-Banach module. Let $I(a)$ be the ideal of S generated by the coefficients of a. It is then clear that one can take $S(a) = S/I(a)$. $\square$

Lemma 3.11. *Let $a \in \widetilde{\mathcal{R}}^r_{\widehat{K^{\mathrm{ur}}}} \widehat{\otimes}_{\mathbb{Q}_p} S$. Then there exists an analytic subspace $M(S(a, r))$ of $M(S)$ such that for any map $g : S \to R$ of $\mathbb{Q}_p$-affinoid algebras, $g(a) \in (K \otimes_{K_0} \mathbf{B}^{\dagger,\rho(r)}_{\mathrm{rig},K}) \widehat{\otimes}_{\mathbb{Q}_p} R$ if and only if the map $M(R) \to M(S)$ factors through $M(S(a, r))$.*

Proof. Note that $\mathbf{B}^{\dagger,\rho(r)}_{\mathrm{rig},K}$ may be identified with the intersections of all $\mathbf{B}^{[\rho(r),\rho(s)]}_K$'s with $0 < s < r$ (one way to see this fact is to use the identification of $\mathbf{B}^{\dagger,\rho(r)}_{\mathrm{rig},K}$ with

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[7]See [23, §5] for a uniform treatment about relative extended Robba rings associated to analytic fields of characteristic p.

[8]The completion is taken with respect to the u-adic topology.

the Robba ring $\mathcal{R}^r_{K'_0}$). To prove the lemma, it then suffices to show that for any $0 < s < r$, there exists an analytic subspace $M(S(a, s, r))$ of $M(S)$ such that for any map $g : S \to R$ of $\mathbb{Q}_p$-affinoid algebras, $g(a) \in (K \otimes_{K_0} \mathbf{B}_K^{[\rho(r),\rho(s)]}) \widehat{\otimes}_{\mathbb{Q}_p} R$ if and only if the map $M(R) \to M(S)$ factors through $M(S(a, s, r))$; we then take $M(S(a, r))$ to be the intersections of all $M(S(a, s, r))$'s.

Let $\widetilde{\mathbf{B}}^{[\rho(r),\rho(s)]}$ be the completion of $\widetilde{\mathbf{B}}^{\dagger,\rho(r)}_{\mathrm{rig}}$ with respect to $\max\{\mathrm{val}^{(0,\rho(r))},$ $\mathrm{val}^{(0,\rho(s))}\}$. One may therefore identify $\mathbf{B}_K^{[\rho(r),\rho(s)]}$ as a closed subspace of $\widetilde{\mathbf{B}}^{[\rho(r),\rho(s)]}$. On the other hand, the closed embedding $\widetilde{\mathbf{B}}^{\dagger,\rho(r)}_{\mathrm{rig}} \hookrightarrow \widetilde{\mathcal{R}}^r_{\widehat{\mathbb{Q}^{\mathrm{ur}}_p}}$ of $\mathbb{Q}_p$-Fréchet algebras induces a closed embedding $\widetilde{\mathbf{B}}^{[\rho(r),\rho(s)]} \hookrightarrow \widetilde{\mathcal{R}}^{[s,r]}_{\widehat{\mathbb{Q}^{\mathrm{ur}}_p}}$ of $\mathbb{Q}_p$-Banach spaces. Therefore, by Hahn–Banach theorem for Banach spaces over discretely valued fields, we deduce that there exists a closed $\mathbb{Q}_p$-subspace V of $\widetilde{\mathcal{R}}^{[s,r]}_{\widehat{K^{\mathrm{ur}}}}$ so that $\widetilde{\mathcal{R}}^{[s,r]}_{\widehat{K^{\mathrm{ur}}}} \cong K \otimes_{K_0} \mathbf{B}_K^{[\rho(r),\rho(s)]} \oplus V$. Hence we have

$$\widetilde{\mathcal{R}}^{[s,r]}_{\widehat{K^{\mathrm{ur}}}} \widehat{\otimes}_{\mathbb{Q}_p} S \cong (K \otimes_{K_0} \mathbf{B}_K^{[\rho(r),\rho(s)]}) \widehat{\otimes}_{\mathbb{Q}_p} S \oplus V \widehat{\otimes}_{\mathbb{Q}_p} S$$

and

$$\widetilde{\mathcal{R}}^{[s,r]}_{\widehat{K^{\mathrm{ur}}}} \widehat{\otimes}_{\mathbb{Q}_p} R \cong (K \otimes_{K_0} \mathbf{B}_K^{[\rho(r),\rho(s)]}) \widehat{\otimes}_{\mathbb{Q}_p} R \oplus V \widehat{\otimes}_{\mathbb{Q}_p} R.$$

Write $a = a_1 + a_2$ with $a_1 \in (K \otimes_{K_0} \mathbf{B}_K^{[\rho(r),\rho(s)]}) \widehat{\otimes}_{\mathbb{Q}_p} S$ and $a_2 \in V \widehat{\otimes}_{\mathbb{Q}_p} S$. It is then obvious that $g(a) \in (K \otimes_{K_0} \mathbf{B}_K^{[\rho(r),\rho(s)]}) \widehat{\otimes}_{\mathbb{Q}_p} R$ if and only if $g(a_2) = 0$. By Proposition 3.5, we may regard a_2 as an element of $\widetilde{\mathcal{R}}^{[s,r]}_{S\widehat{_{K^{\mathrm{ur}}}}}$; then $g(a_2) = 0$ in $\widetilde{\mathcal{R}}^{[s,r]}_{\widehat{K^{\mathrm{ur}}}} \widehat{\otimes}_{\mathbb{Q}_p} R$ if and only if $g(a_2) = 0$ in $\widetilde{\mathcal{R}}^{[s,r]}_{R\widehat{_{K^{\mathrm{ur}}}}}$. Write $a_2 = \sum_{i \in \mathbb{Q}} c_i u^i$ with $c_i \in S_{\widehat{K^{\mathrm{ur}}}}$. Let

$$I(a, s, r) = \sum_{i \in \mathbb{Q}} I(c_i)$$

where $I(c_i)$ is the ideal defined in the proof of Lemma 3.10. It is then clear that one can take $S(a, s, r) = S/I(a, s, r)$. $\square$

4. Construction of finite slope subspaces

Throughout this section, let X, V_X and α be as in §1.1. For any morphism $X' \to X$ of rigid analytic spaces over $\mathbb{Q}_p$, we denote by $V_{X'}$ the pullback of V_X on X' which is a locally free coherent $\mathcal{O}_{X'}$-module of rank d with a continuous $\mathcal{O}_{X'}$-linear G_K-action. In the case when $X = M(S)$ is an affinoid space, we denote V_X by V_S instead. We have defined finite slope subspaces of X with respect to (α, V_X) in Definition 1.1. The goal of this section is to prove that X has a unique finite slope subspace (which may well be empty).

4.1. Prelude.

Proposition 4.1. *The formation of X_{fs} commutes with flat base change. Namely, if $h : X' \to X$ is a flat morphism of separated and reduced rigid analytic spaces over $\mathbb{Q}_p$, and if X_{fs} is a finite slope subspace of X with respect to (α, V_X), then the base change X'_{fs} of X_{fs} via h is a finite slope subspace of X' with respect to $(h^*(\alpha), V_{X'})$.*

Proof. Note that the Sen polynomial for $V_{X'}$ is $Th^*(Q(T))$. By Definition 1.1(1), we have that $Q(j)_\tau$ is a nonzero divisor in X_{fs} for every integer $j \leq 0$ and $\tau \in H_K$. The flatness of h then implies that $h^*(Q(j)_\tau)$ is a nonzero divisor in X'_{fs}. Hence X'_{fs} satisfies (1) of Definition 1.1. Now let $g : M(R) \to X'$ be a map of rigid spaces over $\mathbb{Q}_p$ which factors through $X'_{Q(j)_\tau}$ for every integer $j \leq 0$ and $\tau \in H_K$. Then $h \circ g$ factors through $X_{Q(j)_\tau}$ for every integer $j \leq 0$ and $\tau \in H_K$. By the universal property of X_{fs}, we know that for n sufficiently large, the natural map

$$(K \otimes_{K_0} D^\dagger_{\mathrm{rig}}(V_R))^{\varphi^f = g^*(h^*(\alpha)), \Gamma = 1} \to (D^{+,fn}_{\mathrm{dif}}(V_R))^\Gamma$$

is an isomorphism if and only if $h \circ g$ factors through X_{fs}, i.e. if and only if g factors through X'_{fs}. This implies that X'_{fs} satisfies (2) of Definition 1.1. $\qquad \square$

Proposition 4.2. *There exists at most one finite slope subspace of X.*

Proof. Suppose that X_1, X_2 are two finite slope subspaces of X. Let $\{U_j\}_{j \in J}$ be an admissible affinoid covering of X by affinoid subdomains. It suffices to show that for any $j \in J$, the restrictions of X_1, X_2 on U_j coincide. By Proposition 4.1, we see that the restrictions of X_1, X_2 on U_j are finite slope subspaces of U_j. Thus it reduces to the case that $X = M(S)$ is an affinoid space. We prove this by using Kisin's argument ([25, (5.8)]). Let $I_1, I_2 \subset S$ be the ideals corresponding to X_1, X_2 respectively. Let W be the support of $(I_1 + I_2)/I_1$ in X_1 (with its reduced structure). Let $x \in X_1$ be a closed point. If $x \in X_{Q(j)_\tau}$ for every integer $j \leq 0$ and $\tau \in H_K$, applying (2) of Definition 1.1 to any finite length quotient R of $\mathcal{O}_{X_1,x}$, we get that $x \in X_2$ and $\widehat{\mathcal{O}_{X_1,x}} = \widehat{\mathcal{O}_{X_2,x}}$. This implies that $x \notin W$. Hence, for any $w \in W$ there exists integer $j \leq 0$ and $\tau \in H_K$ such that $Q(j)_\tau(w) = 0$. If W_0 is an irreducible component of W, by [25, Lemma (5.7)], we deduce that there exists $j_{W_0} \leq 0$ and $\tau_{W_0} \in H_K$ such that $Q(j_{W_0})_{\tau_{W_0}}$ vanishes in W_0. It follows that $X_1 \backslash W$ contains $\cap_{W_0 \subseteq W}(X_1)_{Q(j_{W_0})_{\tau_{W_0}}}$. The latter is Zariski open and dense in X_1 since W has only finitely many components. A fortiori we see that $X_1 \backslash W$, which is contained in X_2, is Zariski open and dense in X_1, yielding $X_1 \subset X_2$. Thus $X_2 = X_1$. $\qquad \square$

Remark 4.3. The proof of Proposition 4.2 actually implies that there exists at most one analytic subspace of X which satisfies Definition 1.1(1) and Definition 1.1(2) for all finite $\mathbb{Q}_p$-algebras R.

Proposition 4.4. *Let $\{U_j\}_{j \in J}$ be an admissible covering of X by affinoid sub-domains. Suppose that each U_j has the finite slope subspace $(U_j)_{fs}$. Then $\{(U_j)_{fs}\}_{j \in J}$ glues to form the finite slope subspace of X.*

Proof. By the uniqueness of finite slope subspaces, we see that $\{(U_j)_{fs}\}_{j \in J}$ glues to form an analytic subspace X_{fs} of X. It is then clear that X_{fs} satisfies (1) of Definition 1.1. Now let $g : M(R) \to X$ be a morphism of rigid analytic spaces over $\mathbb{Q}_p$ which factors through $X_{Q(j)}$ for each integer $j \leq 0$. The pullback $\{g^{-1}(U_j)\}$ forms an admissible covering of $M(R)$. We choose a finite covering $\{M(R_i)\}_{i \in I}$ of $M(R)$ by affinoid subdomains which refines $\{g^{-1}(U_j)\}$. It then follows that for each $i \in I$, the natural map

$$(K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_{R_i}))^{\varphi^f = g^*(\alpha), \Gamma = 1} \to \mathrm{D}_{\mathrm{dif}}^{+, fn}(V_{R_i})^{\Gamma}$$

is an isomorphism for all sufficiently large n. We deduce from Propositions 2.18 and 2.9 that the natural map

$$(K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_R))^{\varphi^f = g^*(\alpha), \Gamma = 1} \to \mathrm{D}_{\mathrm{dif}}^{+, fn}(V_R)^{\Gamma}$$

is an isomorphism for all sufficiently large n. This yields that X_{fs} is the finite slope subspace of X. $\square$

4.2. Techniques.
We start by introducing some notations. For an affinoid algebra S, $\alpha \in S^{\times}$, and a as in Lemma 3.8, using Lemma 3.10, we denote by $M(S(\alpha, a))$ the intersection of

$$M\left(S\left(\sum_{m \in \mathbb{Z}} \alpha^{-(m+1)} \varphi^m(a_{iq^{-m}})\right)\right)$$

for all rational numbers $i < 0$. From now on, let V_S be a locally free S-linear representations of G_K of rank d.

Proposition 4.5. *Let $\alpha \in S^{\times}$, and let $\beta \in (K \otimes_{K_0} \mathbf{B}_K^{\dagger, s})^{\times}$ satisfying $|\beta| > |\alpha^{-1}|$ (here $|\beta|$ denotes the p-adic norm of β in $K \otimes_{K_0} \mathbf{B}_K^{\dagger, s}$). Then for any $a \in K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^{\dagger, s}(V_S)$ there exists an E-analytic subspace $M(S(\alpha, \beta, a))$ of $M(S)$ such that for any morphism $g : S \to R$ of affinoid algebras over E, the equation*

$$\varphi^f(b) - \beta g(\alpha) b = g(a) \tag{4.1}$$

has a solution $b \in K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^{\dagger, s}(V_R)$ if and only if the map $M(R) \to M(S)$ factors through $M(S(\alpha, \beta, a))$. Furthermore, the solution b is unique in this case.

Proof. Granting the assertion of the proposition, it is then clear that the construction of $M(S(\alpha, \beta, a))$ is compatible with base change. Thus it suffices to prove the proposition for each affinoid subdomain of an affinoid covering of $M(S)$. Therefore

it reduces to the case that V_S is free over S. Choose an S-basis $e_1, \ldots, e_d$ of V_S, and write $a = \sum_{i=1}^{d} a_i e_i$ with $a_i \in K \otimes_{K_0} (\mathbf{B}_{\mathrm{rig}}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S)$. Since φ acts trivially on V_R, (4.1) admits a solution in $V_R \otimes_R (\widetilde{\mathcal{R}}_{\widehat{K^{\mathrm{ur}}}}^{\rho(s)} \widehat{\otimes}_{\mathbb{Q}_p} R)$ if and only if each Frobenius equation

$$\varphi^f(b_i) - \beta g(\alpha) b_i = g(a_i) \tag{4.2}$$

admits a solution b_i in $\widetilde{\mathcal{R}}_{\widehat{K^{\mathrm{ur}}}}^{\rho(s)} \widehat{\otimes}_{\mathbb{Q}_p} R$. By [20, Proposition 3.3.2][9], we may choose some

$$x \in (K \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger})^{\times}$$

such that $y = \beta \varphi^f(x)/x$ belongs to K. Using Frobenius, we see that x actually lies in $(K \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger,s})^{\times}$. We thus rewrite (4.2) as

$$\varphi^f(xb_i) - yg(\alpha)xb_i = \varphi^f(x)g(a_i).$$

Note that $|y| = |\beta|$. Thus $|y^{-1}g(\alpha)^{-1}| \leq |\beta^{-1}||\alpha^{-1}| < 1$. We deduce from Lemma 3.8 that (4.1) admits a solution in $V_R \otimes_R (\widetilde{\mathcal{R}}_{\widehat{K^{\mathrm{ur}}}}^{\rho(s)} \widehat{\otimes}_{\mathbb{Q}_p} R)$ if and only if $M(R) \to M(S)$ factors through

$$M(S') = \bigcap_{1 \leq i \leq d} M(y\alpha, \varphi^f(x)a_i).$$

Furthermore, in this case, the solution is unique. Let b be the solution of (4.1) in $V_{S'} \otimes_{S'} (\widetilde{\mathcal{R}}_{\widehat{K^{\mathrm{ur}}}}^{\rho(s)} \widehat{\otimes}_{\mathbb{Q}_p} S')$. Let L be a finite extension of K so that $\mathrm{D}_{\mathrm{rig},L}^{\dagger,s}(V_S)$ is free over $\mathbf{B}_{\mathrm{rig},L}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S$. Choose a $\mathbf{B}_{\mathrm{rig},L}^{\dagger,s} \widehat{\otimes}_{\mathbb{Q}_p} S'$-basis $\{f_1, \ldots, f_d\}$ of $\mathrm{D}_{\mathrm{rig},L}^{\dagger,s}(V_{S'})$. Since

$$(K \otimes_{K_0} \mathrm{D}_{\mathrm{rig},L}^{\dagger,s}(V_{S'})) \otimes_{(K \otimes_{K_0} \mathbf{B}_{\mathrm{rig},L}^{\dagger,s}) \widehat{\otimes}_{\mathbb{Q}_p} S'} (\widetilde{\mathcal{R}}_{\widehat{K^{\mathrm{ur}}}}^{\rho(s)} \widehat{\otimes}_{\mathbb{Q}_p} S') = V_{S'} \otimes_{S'} (\widetilde{\mathcal{R}}_{\widehat{K^{\mathrm{ur}}}}^{\rho(s)} \widehat{\otimes}_{\mathbb{Q}_p} S'),$$

we may write $b = \sum_{i=1}^{d} b_i f_i$ with $b_i \in \widetilde{\mathcal{R}}_{\widehat{K^{\mathrm{ur}}}}^{\rho(s)} \widehat{\otimes}_{\mathbb{Q}_p} S'$. Let S'' be the $\mathbb{Q}_p$-affinoid algebra defined by

$$M(S'') = \bigcap_{1 \leq i \leq d} M(S'(c_i, s))$$

(see Lemma 3.11 for the definition of $M(S'(c_i, s))$). By Lemma 3.11, $g(b)$ belongs to $K \otimes_{K_0} \mathrm{D}_{\mathrm{rig},L}^{\dagger,s}(V_R)$ if and only if the map $M(R) \to M(S')$ factors through $M(S'')$. Furthermore, by the uniqueness of the solution of (4.1), the image of b in $K \otimes_{K_0} \mathrm{D}_{\mathrm{rig},L}^{\dagger,s}(V_{S''})$ is H_K-invariant; hence it is in $K \otimes_{K_0} \mathrm{D}_{\mathrm{rig},K}^{\dagger,s}(V_{S''})$ by Theorem 2.4(4). Therefore we can take $S(\alpha, \beta, a) = S''$.   □

By Lemma 2.11, $(\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S))^{\varphi^f = \alpha}$ is contained in $\mathrm{D}_{\mathrm{rig}}^{\dagger,s}(V_S)$ for any $\alpha \in S$ and $s \geq s(V_S)$. Thus for any $n \geq n(V_S)$, we have a natural map

$$\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S)^{\varphi^f = \alpha} \to \mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)$$

via the localization map ι_n.

---

[9]By the definition of $\Gamma_{\mathrm{con}}^{\mathrm{alg}}$, one may identify it with $K \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger}$.

Proposition 4.6. *Let $\alpha \in S^{\times}$. Then for any $k > \log_{|\pi_K^{-1}|} |\alpha^{-1}|$ and $n \geq n(V_S)$, the natural map*

$$\iota_{n,K} : K \otimes_{K_0} D_{\mathrm{rig}}^{\dagger}(V_S)^{\varphi^f = \alpha} \to D_{\mathrm{dif}}^{+,fn}(V_S)/(t^k)$$

is injective.

Proof. Let $a \in (K \otimes_{K_0} D_{\mathrm{rig}}^{\dagger}(V_S))^{\varphi^f = \alpha}$, and let a_m be its image in $D_{\mathrm{dif}}^{+,fm}(V_S)/(t^k)$ via $\iota_{m,k}$ for any $m \geq n(V_S)$. The relation $\varphi^f(a) = \alpha a$ and Proposition 2.15(3) imply

$$a_m = \alpha^{n-m} a_n.$$

Thus if $a_n = 0$, then $a_m = 0$ for all $m \geq n(V_S)$. This implies $t_K^k | a$ by Proposition 2.16. Now suppose that a lies in the kernel of the map, and write $a = t_K^k a'$ for some $a' \in D_{\mathrm{rig}}^{\dagger}(V_S)$. It follows that

$$\varphi^f(a') = (t_K/\varphi^f(t_K))^k \alpha a'. \tag{4.3}$$

By Proposition 2.16 and assumption,

$$|(t_K/\varphi^f(t_K))^k| = |\pi_K^{-1}|^k > |\alpha^{-1}|.$$

Hence $a' = 0$ is the unique solution of (4.3) by Proposition 4.5. $\square$

Proposition 4.7. *For any $n \geq n(V_S)$, $k > \log_{|\pi_K^{-1}|} |\alpha^{-1}|$ and $a \in D_{\mathrm{dif}}^{+,fn}(V_S)/(t^k)$, there exists an E-analytic subspace $M(S(k,\alpha,a))$ of $M(S)$ such that for any map $g : S \to R$ of affinoid algebras over E, $g(a) \in D_{\mathrm{dif}}^{+,fn}(V_R)/(t^k)$ is contained in the image of*

$$\iota_{n,K} : K \otimes_{K_0} D_{\mathrm{rig}}^{\dagger}(V_R)^{\varphi^f = g(\alpha)} \to D_{\mathrm{dif}}^{+,fn}(V_R)/(t^k)$$

if and only if the map $M(R) \to M(S)$ factors through $M(S(k,\alpha,a))$.

Proof. As in the proof of Proposition 4.5, it suffices to treat the case that V_S is free. Using Proposition 2.16, we choose $\tilde{a} \in D_{\mathrm{rig}}^{\dagger,rfn}(V_S)$ such that the image of $\iota_{m,K}(\tilde{a})$ in $D_{\mathrm{dif}}^{+,fm}(V_S)/(t^k)$ is $\alpha^{m-n} a$ for each $m \geq n$. If $g(a)$ can be lifted to $\tilde{b} \in (K \otimes_{K_0} D_{\mathrm{rig}}^{\dagger}(V_R))^{\varphi^f = g(\alpha)}$, it then follows that the image of $\iota_{m,K}(\tilde{b})$ in $D_{\mathrm{dif}}^{+,fm}(V_R)/(t^k)$ is $g(\alpha)^{m-n} g(a)$. Then by Proposition 2.16, we see that t_K^k divides $\tilde{b} - g(\tilde{a})$ in $K \otimes_{K_0} D_{\mathrm{rig}}^{\dagger,rfn}(V_R)$. Hence

$$b = (\tilde{b} - g(\tilde{a}))/t_K^k$$

is a solution of the equation

$$(\varphi^f - g(\alpha))(g(\tilde{a}) + t_K^k b) = 0. \tag{4.4}$$

Conversely, any solution $b \in K \otimes_{K_0} D_{rig}^{\dagger, r_{fn}}(V_R)$ of (4.4) gives rise to the desired lift $g(\tilde{a}) + t_K^k b$ of $g(a)$. Therefore, we conclude that $g(a)$ can be lifted to $(K \otimes_{K_0} D_{rig}^{\dagger}(V_R))^{\varphi^f = g(\alpha)}$ if and only if (4.4) has a solution $b \in K \otimes_{K_0} D_{rig}^{\dagger, r_{fn}}(V_R)$.

A short computation shows that (4.4) can be rewritten as

$$\varphi^f(t_K)^k (\varphi^f - (t_K/\varphi^f(t_K))^k g(\alpha)) b = (g(\alpha) - \varphi^f)(g(\tilde{a})).$$

By the construction of $\tilde{a}$, t_K^k divides $\varphi^f(\tilde{a}) - \alpha \tilde{a}$ in $K \otimes_{K_0} D_{rig}^{\dagger, r_{fn}}(V_S)$. Note that

$$(t_K) = (\varphi^f(t_K))$$

in $K \otimes_{K_0} B_{rig,K}^{\dagger, r_{f(n+1)}}$ by Proposition 2.16(3). Hence $\varphi^f(t_K)^k$ divides $(g(\alpha) - \varphi^f(\alpha))(g(\tilde{a}))$ in $K \otimes_{K_0} D_{rig}^{\dagger, r_{f(n+1)}}(V_R)$. We therefore deduce that (4.4) has a solution in $D_{rig}^{\dagger, r_{fn}}(V_R)$ if and only if the equation

$$(\varphi^f - (t_K/\varphi^f(t_K))^k g(\alpha)) b = \varphi^f(t_K)^{-k}(g(\alpha) - \varphi^f)(g(\tilde{a})). \qquad (4.5)$$

has a solution in $K \otimes_{K_0} D_{rig}^{\dagger, r_{f(n+1)}}(V_R)$. In fact, if b is such a solution, we have

$$b \in K \otimes_{K_0} D_{rig}^{\dagger, r_{fn}}(V_R)$$

by Lemma 2.11. The assumption implies that $|(t_K/\varphi^f(t_K))^k| = |\pi_K^{-1}|^k > |\alpha^{-1}|$. Using Proposition 4.5, $g(a)$ can be lifted to $(K \otimes_{K_0} D_{rig}^{\dagger}(V_R))^{\varphi^f = g(\alpha)}$ if and only if the map $M(R) \to M(S)$ factors through

$$M(S(\alpha, (t_K/\varphi^f(t_K))^k, \varphi^f(t_K)^{-k}(\alpha - \varphi^f)(\tilde{a})).$$

Thus we can take $S(k, \alpha, a)$ to be $S(\alpha, (t_K/\varphi^f(t_K))^k, \varphi^f(t_K)^{-k}(\alpha - \varphi^f)(\tilde{a}))$. $\quad\square$

Corollary 4.8. *For any integer $n \geq n(V_S)$ and positive integer $k > \log_{|\pi_K^{-1}|} |\alpha^{-1}|$, there exists an E-analytic subspace $M(S(k, \alpha, n))$ of $M(S)$ such that for any map $g : S \to R$ of affinoid algebras over E, the R-submodule $g((D_{dif}^{+, fn}(V_S)/(t^k))^\Gamma)$ of $(D_{dif}^{+, fn}(V_R)/(t^k))^\Gamma$ is contained in the image of*

$$(K \otimes_{K_0} D_{rig}^{\dagger}(V_R))^{\varphi^f = g(\alpha), \Gamma = 1} \to (D_{dif}^{+, fn}(V_R)/(t^k))^\Gamma$$

if and only if the map $M(R) \to M(S)$ factors through $M(S(k, \alpha, n))$.

Proof. Recall that by Proposition 4.6, the map

$$K \otimes_{K_0} D_{rig}^{\dagger}(V_R)^{\varphi^f = g(\alpha)} \to D_{dif}^{+, fn}(V_R)/(t^k)$$

is injective. Since it is also Γ-equivariant, we deduce that for any $a \in (D_{dif}^{+, fn}(V_S)/(t^k))^\Gamma$, if $g(a)$ can be lifted to $K \otimes_{K_0} D_{rig}^{\dagger}(V_R)^{\varphi^f = g(\alpha)}$, then the lift is also Γ-invariant. Thus we can take $M(S(k, \alpha, n))$ to be the intersection of $M(S(k, \alpha, a))$ for all $a \in (D_{dif}^{+, fn}(V_S)/(t^k))^\Gamma$. $\quad\square$

Corollary 4.9. *Keep notations as in Corollary 4.8. Then there exists an E-analytic subspace $M(S(k,\alpha))$ of $M(S)$ such that for any map $g : S \to R$ of affinoid algebras over E, the R-submodule $g((\mathrm{D}_{\mathrm{dif}}^{+,fn}(V_S)/(t^k))^\Gamma)$ of $(\mathrm{D}_{\mathrm{dif}}^{+,fn}(V_R)/(t^k))^\Gamma$ is contained in the image of*

$$(K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^\dagger(V_R))^{\varphi^f = g(\alpha), \Gamma = 1} \to (\mathrm{D}_{\mathrm{dif}}^{+,fn}(V_R)/(t^k))^\Gamma \qquad (4.6)$$

for all sufficiently large n if and only if the map $M(R) \to M(S)$ factors through $M(S(k,\alpha))$.

Proof. It is clear that we can take $M(S(k,\alpha))$ to be the intersection of $M(S(k,\alpha,n))$ for all $n \geq n(V_S)$. $\square$

4.3. Finite slope subspaces.

Theorem 4.10. *The rigid analytic space X has a unique finite slope subspace X_{fs}.*

Proof. By Proposition 4.4, it suffices to treat the case that $X = M(S)$ is an affinoid space. Let

$$X' = \bigcap_{k > \log_{|\pi_K^{-1}|} |\alpha^{-1}|} M(S(k,\alpha)).$$

Now for each $i \geq 1$ and $\tau \in \mathrm{H}_K$, let X_i' be the Zariski closure of

$$\bigcap_{\tau \in \mathrm{H}_K} X' = \bigcap_{\substack{0 \leq j \leq i-1, \\ \tau \in \mathrm{H}_K}} X'_{Q(-j)_\tau}.$$

We claim that

$$X_{fs} = \bigcap_{i \geq 1} X_i'$$

is the finite slope subspace of X. First note that the decreasing sequence of closed subspaces $X_1' \supseteq X_2' \supseteq \ldots$ becomes constant eventually because S is Noetherian. We fix an i_0 such that $X_i' = X_{i_0}'$ for all $i \geq i_0$. Hence $X_{fs} = X_i'$ for all $i \geq i_0$. Thus for any $i \geq i_0$ and $\tau \in \mathrm{H}_K$, we have

$$(X_{fs})_{P(i)_\tau} \supseteq X_{fs} \cap X'_{P(i)_\tau} = X_{i_0} \cap X'_{P(i)_\tau} \supseteq \bigcap_{\tau \in \mathrm{H}_K} X'_{P(i)_\tau}.$$

Therefore the Zariski closure of $(X_{fs})_{P(i)_\tau}$ contains the Zariski closure of $\cap_{\tau \in \mathrm{H}_K} X'_{P(i)_\tau}$, which is $X_i' = X_{fs}$; this yields that X_{fs} satisfies (1) of Definition 1.1.

Now suppose that $g : M(R) \to M(S)$ is a map of affinoid spaces over E which factors through $X_{Q(j)}$ for every $j \leq 0$. It follows from Corollary 2.27 that for each $k \geq 1$ and $n \geq n(V_S)$, the natural map

$$(\mathrm{D}_{\mathrm{dif}}^{+,fn}(V_R))^\Gamma \to (\mathrm{D}_{\mathrm{dif}}^{+,fn}(V_R)/(t^k))^\Gamma$$

is an isomorphism.

Hence (1.1) is an isomorphism if and only if the natural map

$$(K \otimes_{K_0} D_{\mathrm{rig}}^{\dagger}(V_R))^{\varphi^f = g^*(\alpha), \Gamma = 1} \to (D_{\mathrm{dif}}^{+, fn}(V_R)/(t^k))^{\Gamma} \qquad (4.7)$$

is surjective for some (hence any) $k \geq 1$. By Corollary 2.27, the natural map

$$(D_{\mathrm{dif}}^{+, fn}(V_S)/(t^k))^{\Gamma} \otimes_S R \to (D_{\mathrm{dif}}^{+, fn}(V_R)/(t^k))^{\Gamma}$$

is an isomorphism. Hence by Corollary 4.8, the map (4.7) is surjective if and only if the map $g : M(R) \to M(S)$ factors through $M(S(k, \alpha, n))$ for each $k > \log_{|\pi_K^{-1}|} |\alpha^{-1}|$ by Corollary 4.9. We thus conclude that (1.1) is an isomorphism for all sufficiently large n if and only if $g : M(R) \to M(S)$ factors through X_{fs}. This yields that X_{fs} satisfies (2) of Definition 1.1. $\square$

Proposition 4.11. *For any affinoid subdomain $M(S)$ of X_{fs} and $k > \log_{|\pi_K^{-1}|} |\alpha^{-1}|$, we have $S(k, \alpha) = S$. As a consequence, for such k, the natural map*

$$(K \otimes_{K_0} D_{\mathrm{rig}}^{\dagger}(V_S))^{\varphi^f = \alpha, \Gamma = 1} \to (D_{\mathrm{dif}}^{+, fn}(V_S)/(t^k))^{\Gamma}$$

is an isomorphism for all $n \geq n(V_S)$.

Proof. It is obvious that the finite slope subspace of X_{fs} is X_{fs} itself. We then deduce that $(M(S))_{fs} = M(S)$ since the formation of finite slope subspaces commutes with flat base change by Proposition 4.1. This yields that $M(S) \subseteq M(S(k, \alpha))$ following the construction of the finite slope subspace in Theorem 4.10; hence $M(S) = M(S(k, \alpha))$. This yields the surjectivity of the map. The injectivity follows from Proposition 4.6. $\square$

Theorem 4.12. *Let $M(S)$ be an affinoid subdomain of X_{fs}. Then for any $n \geq n(V_S)$ and $k > \log_{|\pi_K^{-1}|} |\alpha^{-1}|_{\mathrm{sp}}$ where the norm is taken in S, the natural map of sheaves*

$$(K \otimes_{K_0} \mathscr{D}_{\mathrm{rig}}^{\dagger}(V_S))^{\varphi^f = \alpha, \Gamma = 1} \to (\mathscr{D}_{\mathrm{dif}}^{+, fn}(V_S)/(t^k))^{\Gamma}$$

is an isomorphism. As a consequence, $(\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_{X_{fs}}))^{\varphi^f = \alpha, \Gamma = 1}$ is a coherent sheaf on X_{fs}.

Proof. It follows from Proposition 4.11 that the map of sheaves

$$(K \otimes_{K_0} \mathscr{D}_{\mathrm{rig}}^{\dagger}(V_S))^{\varphi^f = \alpha, \Gamma = 1} \to (\mathscr{D}_{\mathrm{dif}}^{+, fn}(V_S)/(t^k))^{\Gamma}$$

is an isomorphism. By Proposition 2.21, we therefore deduce that $(\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_S))^{\varphi^f = \alpha, \Gamma = 1}$ is a coherent sheaf. $\square$

Theorem 4.13. *For any E-affinoid algebra R and morphism $g : M(R) \to X_{fs}$ which factors through $X_{Q(j)}$ for every integer $j \leq 0$, the natural map*

$$(K \otimes_{K_0} \mathscr{D}_{\mathrm{rig}}^{\dagger}(V_R))^{\varphi^f = g^*(\alpha), \Gamma = 1} \to (\mathscr{D}_{\mathrm{dif}}^{+, fn}(V_R)/(t^k))^{\Gamma}$$

is an isomorphism for all sufficiently large k. As a consequence, $(\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_R))^{\varphi^f = g^*(\alpha), \Gamma = 1}$ is a coherent sheaf.

Proof. We choose an admissible affinoid covering $\{M(S_i)\}_{i \in I}$ of X_{fs} by affinoid subdomains. Let $\{M(R_j)\}_{j \in J}$ be a finite covering of $M(R)$ which refines the pullback of the covering $\{M(S_i)\}_{i \in I}$ on $M(R)$. Suppose that $M(R_j)$ maps to $M(S_{i_j})$ for each $j \in J$. Let k be a positive integer such that

$$k > \log_{|\pi_K^{-1}|} \max_{i \in I} \{|\alpha^{-1}| \text{ in } S_i\}.$$

Now for any affinoid subdomain $M(R')$ of some $M(R_j)$,

$$(\mathrm{D}_{\mathrm{dif}}^{+, fn}(V_{R'})/(t^k))^{\Gamma = 1} = (\mathrm{D}_{\mathrm{dif}}^{+, fn}(V_{S_{i_j}})/(t^k))^{\Gamma = 1} \otimes_{S_{i_j}} R'$$

by Corollary 2.30 because $M(R')$ maps to $X_{P(k)}$. On the other hand, by Proposition 4.11, we have $M(S_{i_j}(k, \alpha)) = M(S_{i_j})$, yielding that

$$(K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_{R'}))^{\varphi^f = g^*(\alpha), \Gamma = 1} \to (\mathrm{D}_{\mathrm{dif}}^{+, fn}(V_{R'})/(t^k))^{\Gamma}$$

is surjective. Furthermore, it is injective by Proposition 4.6; so it is an isomorphism. Hence

$$(K \otimes_{K_0} \mathscr{D}_{\mathrm{rig}}^{\dagger}(V_{R_j}))^{\varphi^f = g^*(\alpha), \Gamma = 1} \to (\mathscr{D}_{\mathrm{dif}}^{+, fn}(V_{R_j})/(t^k))^{\Gamma}$$

is an isomorphism. This yields the theorem. $\qquad\square$

Remark 4.14. Our finite slope subspace X_{fs} coincides with Nakamura's generalization of Kisin's finite slope subspace [31]. In fact, as noted in Remark 4.3, to characterize our finite slope subspaces, it suffices to test only finite $\mathbb{Q}_p$-algebras R in Definition 1.1(2). By the argument in [25, (5.8)], the same thing holds for Nakamura's finite slope subspaces as well. For such R, we have the comparisons

$$(\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_R))^{\Gamma} = D_{\mathrm{crys}}^+(V_R) \quad \text{and} \quad (\mathrm{D}_{\mathrm{dif}}^+(V_R))^{\Gamma} = D_{\mathrm{dR}}^+(V_R)$$

by [2, Théorème 3.6] and [17, Théorème 3.9] respectively. Thus our Definition 1.1(2) coincides with the counterpart of Nakamura's in this case; hence the claim.

5. Global triangulation of refined families

In this section, we prove the main results of this paper.

5.1. Weakly refined families.

5.1. Weakly refined families. From now on, let X be a reduced rigid analytic space over E, and let V_X be a family of weakly refined p-adic representations of G_K of dimension d on X as in §1.3. We further suppose $\kappa_1 = 0$. Therefore the Sen polynomial for V_X is of the form $TQ(T)$ with $Q(T) \in K \otimes_{\mathbb{Q}_p} \mathcal{O}(X)[T]$. As in §4, we put $P(i) = \prod_{j=0}^{i-1} Q(-j)$ for $i \geq 1$.

Convention 5.1. For $c \in \mathbb{R}$, $1 \leq i \leq d$, $x \in X$ and $? \in \{>, <, \leq, \geq\}$, we say $\kappa_i(x)?c$ if $\kappa_i(x)_\tau?c$ for any $\tau \in H_K$.

The first goal of this subsection is to show that the finite slope subspace of X with respect to the pair (F, V_X) is X itself (Theorem 5.3). We start by collecting some basic properties about the p-adic representations V_z for $z \in Z$.

Lemma 5.2. *The following are true.*

(1) *If $x \in X_{P(k)}$ for some $k \geq 1$, then*

$$\dim_{k(x)}(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_x)/(t^k))_\tau^\Gamma \leq 1$$

for any $\tau \in H_K$.

(2) *For any $z \in Z$ and $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, $\dim_{k(z)}(\mathrm{D}_{\mathrm{rig}}^\dagger(V_z))_\sigma^{\varphi^f = F(z), \Gamma = 1} \geq 1$.*

(3) *For any $z \in Z$, if $v_K(F(z)) < -\kappa_i(z)$ for all $i \geq 2$, then $\mathrm{D}_{\mathrm{rig}}^\dagger(V_z)_\sigma^{\varphi^f = F(z), \Gamma = 1}$ has dimension 1 over $k(z)$. Furthermore, for any $k \geq 1$ satisfying $v_K(F(z)) < k \leq -\kappa_i(z)$ for all $i \geq 2$, the natural map*

$$(K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^\dagger(V_z)_\sigma)^{\varphi^f = F(z), \Gamma = 1} \to \oplus_{\tau \in H_\sigma}(\mathrm{D}_{\mathrm{dif}}^{+,fn}(V_z)/(t^k))_\tau^\Gamma \qquad (5.1)$$

is an isomorphism.

Proof. By Corollary 2.27, the map

$$(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_x)/(t^k))^\Gamma \to (\mathrm{D}_{\mathrm{Sen}}^n(V_x))^\Gamma$$

is an isomorphism. On the other hand, since $x \in X_{P(k)}$, we see that $\kappa_1(x) = 0$ is a multiplicity-one root of the Sen polynomial for V_x. This implies

$$\dim_{k(x)}(\mathrm{D}_{\mathrm{Sen}}^n(V_x))_\tau^\Gamma \leq 1$$

for any $\tau \in H_K$. Hence $\dim_{k(x)}(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_x)/(t^k))_\tau^\Gamma \leq 1$, yielding (1). For (2), note that the Hodge–Tate weights of V_z are all nonpositive. Hence by Berger's dictionary ([2, Théorème 3.6]),

$$\dim_{k(z)}(\mathrm{D}_{\mathrm{rig}}^\dagger(V_z))_\sigma^{\varphi^f = F(z), \Gamma = 1} = \dim_{k(z)}(D_{\mathrm{crys}}(V_z))_\sigma^{\varphi^f = F(z)}$$

$$\geq \dim_{k(z)}(D_{F(z)})_\sigma^{\varphi^f = F(z)} = 1,$$

where the inequality follows from Definition 1.5(d).

For (3), since $k > v_K(F(z))$, we get the injectivity of (5.1) by Proposition 4.6. On the other hand, note that $z \in X_{P(k)}$ since $k \leq -k_i(z)$ for $2 \leq i \leq d$. It follows by (1) that $\dim(\mathrm{D}_{\mathrm{dif}}^{+,fn}(V_z)/(t^k))_\tau^\Gamma \leq 1$. Hence the dimension of right hand side of (5.1) over $k(x)$ is at most $[K : \mathbb{Q}_p]/f$. On the other hand, by (2), the left hand side of (5.1) has $k(x)$-dimension at least $[K : \mathbb{Q}_p]/f$. Putting everything together, we conclude (5.1) is an isomorphism, and

$$\dim_{k(z)}(\mathrm{D}_{\mathrm{rig}}^\dagger(V_z))_\sigma^{\varphi^f=F(z),\Gamma=1} = 1. \qquad \square$$

Theorem 5.3. *The finite slope subspace of X with respect to (F, V_X) is X itself.*

Proof. Since Z is Zariski dense in X, it suffices to prove $Z \subset X_{fs}$. Now let $z \in Z$, and let $M(S)$ be an affinoid subdomain containing z. Let k be an integer such that

$$k > \log_{|\pi_K^{-1}|} |F^{-1}|,$$

where $|\cdot|$ means the sup norm on $M(S)$, i.e. the spectral norm of S. It follows that

$$k > \log_{|\pi_K^{-1}|} |F(x)^{-1}|$$

for any $x \in M(S)$. Now for $z' \in Z_k \cap M(S)$, we first have

$$-k_i(z') > k > v_K(F(z')) = \log_{|\pi_K^{-1}|} |F(z')^{-1}|$$

for $i \geq 2$ by the definition of Z_k. We then deduce that the natural map

$$(K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^\dagger(V_{z'}))^{\varphi^f=F(z'),\Gamma=1} \to (\mathrm{D}_{\mathrm{dif}}^{+,fn}(V_{z'})/(t^k))^\Gamma$$

is an isomorphism by Lemma 5.2(3). Hence $z' \in M(S(k,F))$ by Corollary 4.8. Since $Z_k \cap M(S)$ is Zariski dense in $M(S)$ by Definition 1.5(e) and $M(S(k,F))$ is Zariski closed by its construction, we conclude $S(k,F) = S$ for all $k > \log_{|\pi_K^{-1}|} |F^{-1}|$. Furthermore, for any $i \geq 1$, since $Z_i \cap M(S)$ is Zariski dense in $M(S)$, we deduce that $M(S)_{P(i)} \supset Z_i \cap M(S)$ is also Zariski dense in $M(S)$. We therefore conclude $M(S)_{fs} = M(S)$ following the construction of finite slope subspace. Hence $z \in M(S)_{fs} \subset X_{fs}$. $\qquad \square$

The following theorem follows immediately from Theorem 5.3 and Theorem 4.12.

Theorem 5.4. *Let $M(S)$ be an affinoid subdomain of X. Then for any $k > \log_{|\pi_K^{-1}|} |F^{-1}|$,*

$$(K \otimes_{K_0} \mathscr{D}_{\mathrm{rig}}^\dagger(V_S))^{\varphi^f=F,\Gamma=1} \to (\mathscr{D}_{\mathrm{dif}}^{+,fn}(V_S)/(t^k))^\Gamma$$

is an isomorphism. As a consequence, $(\mathscr{D}_{\mathrm{rig}}^\dagger(V_X))^{\varphi^f=F,\Gamma=1}$ is a coherent sheaf on X.

Our next goal is to show that the saturated locus X_s of V_X (see Definition 1.9) is Zariski open and dense. To do this, we need to investigate the specialization maps

$$\mathscr{D}^\dagger_{\mathrm{rig}}(V_X)^{\varphi^f=F,\Gamma=1} \otimes_{\mathcal{O}_X} k(x) \to D^\dagger_{\mathrm{rig}}(V_x)^{\varphi^f=F(x),\Gamma=1}$$

for $x \in X$.

Proposition 5.5. *Let $M(S)$ be an affinoid subdomain of X, and let k be a positive integer satisfying $k > \log_{|\pi_K^{-1}|} |F^{-1}|$. Then the following are true.*

(1) *For any $x \in M(S)_{P(k)}$, the natural map*

$$(K \otimes_{K_0} D^\dagger_{\mathrm{rig}}(V_x))^{\varphi^f=F(x),\Gamma=1} \to (D^{+,fn}_{\mathrm{dif}}(V_x)/(t^k))^\Gamma \qquad (5.2)$$

is an isomorphism.

(2) *For any $x \in M(S)_{P(k)}$, the natural map*

$$D^\dagger_{\mathrm{rig}}(V_S)^{\varphi^f=F,\Gamma=1} \otimes_S k(x) \to D^\dagger_{\mathrm{rig}}(V_x)^{\varphi^f=F(x),\Gamma=1} \qquad (5.3)$$

is an isomorphism.

(3) *For any $x \in M(S)$,*

$$\dim_{k(x)}(D^\dagger_{\mathrm{rig}}(V_S)^{\varphi^f=F,\Gamma=1}_\sigma \otimes_S k(x)) \geq 1.$$

(4) *For any $x \in M(S)_{P(k)}$,*

$$\dim_{k(x)}(D^\dagger_{\mathrm{rig}}(V_x))^{\varphi^f=F(x),\Gamma=1}_\sigma = \dim_{k(x)}(D^{+,fn}_{\mathrm{dif}}(V_x)/(t^k))^\Gamma_\tau = 1.$$

Proof. Let $x \in M(S)_{P(k)}$. Consider the following commutative diagram

$$
\begin{array}{ccc}
(K \otimes_{K_0} D^\dagger_{\mathrm{rig}}(V_S))^{\varphi^f=F,\Gamma=1} \otimes_S k(x) & \longrightarrow & (D^{+,fn}_{\mathrm{dif}}(V_S)/(t^k))^\Gamma \otimes_S k(x) \\
\downarrow & & \downarrow \\
(K \otimes_{K_0} D^\dagger_{\mathrm{rig}}(V_x))^{\varphi^f=F(x),\Gamma=1} & \longrightarrow & (D^{+,fn}_{\mathrm{dif}}(V_x)/(t^k))^\Gamma.
\end{array}
$$

The upper horizontal map is an isomorphism by Theorem 5.4. The right vertical map is an isomorphism by Corollary 2.30. The lower horizontal map is injective by Proposition 4.6. We thus deduce that the lower horizontal map and left vertical map are all isomorphisms. This yields (1) and (2).

We first prove (3) for $x \in Z$. In fact, by (2) and Lemma 5.2(3), we have

$$\dim_{k(z)}(D_{\text{rig}}^{\dagger}(V_S)_{\sigma}^{\varphi^f = F, \Gamma = 1} \otimes_S k(z)) = 1$$

for any $z \in Z_k \cap M(S)$. Since $Z_k \cap M(S)$ is Zariski dense in $M(S)$ and $D_{\text{rig}}^{\dagger}(V_S)_{\sigma}^{\varphi^f = F, \Gamma = 1}$ is a finitely generated S-module, we therefore deduce that

$$\dim_{k(x)}(D_{\text{rig}}^{\dagger}(V_S)_{\sigma}^{\varphi^f = F, \Gamma = 1} \otimes_S k(x)) \geq 1$$

for any $x \in M(S)$. For (4), on one hand, we have

$$\dim_{k(x)}(D_{\text{dif}}^{+, fn}(V_x)/(t^k))_{\tau}^{\Gamma} \leq 1$$

by Lemma 5.2(1). On the other hand, we have

$$\dim_{k(x)}(D_{\text{rig}}^{\dagger}(V_x))_{\sigma}^{\varphi^f = F(x), \Gamma = 1} \geq 1$$

by (2) and (3). We then deduce (4) from (1). $\qquad\qquad\qquad\square$

Proposition 5.6. *The subset of saturated points X_s is Zariski open in X.*

Proof. For each $\tau \in H_K$, let Y_τ be the set of $x \in X$ such that the image of the composite

$$\mathscr{D}_{\text{rig}}^{\dagger}(V_X)^{\varphi^f = F, \Gamma = 1} \to D_{\text{rig}}^{\dagger}(V_x)^{\varphi^f = F(x), \Gamma = 1} \to D_{\text{Sen}}^{fn}(V_x) \to D_{\text{Sen}}^{fn}(V_x)_\tau$$

is zero for some (hence all sufficiently large) n. It is clear that each Y_τ is a Zariski closed subset of X. By Proposition 5.5(3), the condition (1) of Definition 1.9 cuts out a Zariski open subset X' of X. For $x \in X'$, by Lemma 5.27 (this is not circular!) we see that x satisfies Definition 1.9(2) if and only if $x \notin Y_\tau$ for any $\tau \in H_K$. Therefore, we conclude

$$X_s = X' \setminus \cup_{\tau \in H_K} Y_\tau$$

is a Zariski open subspace of X. $\qquad\qquad\qquad\square$

Proposition 5.7. *For $x \in X$ and $k > v_K(F(x))$, if $x \in X_{P(k)}$, then $x \in X_s$ and*

$$\dim_{k(x)}(D_{\text{rig}}^{\dagger}(V_x))_{\sigma}^{\varphi^f = F(x), \Gamma = 1} = 1.$$

Proof. Since $k > v_K(F(x))$, we may choose an affinoid neighborhood $M(S)$ of x such that $k > \log_{|\pi_K^{-1}|} |F^{-1}|$ in S. By Proposition 5.5 (2) and (4), we first deduce that x satisfies Definition 1.9(1), and

$$\dim_{k(x)}(D_{\text{rig}}^{\dagger}(V_x))_{\sigma}^{\varphi^f = F(x), \Gamma = 1} = 1.$$

Note that

$$(\mathrm{D}_{\mathrm{dif}}^{+,fn}(V_x)/(t^k))^{\Gamma} \to (\mathrm{D}_{\mathrm{Sen}}^{fn}(V_x))^{\Gamma}$$

is an isomorphism by Corollary 2.30. Thus by Proposition 5.5 (1) and (2), we deduce that

$$(K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S))^{\varphi^f = F, \Gamma = 1} \otimes_S k(x) \to (\mathrm{D}_{\mathrm{Sen}}^{fn}(V_x))^{\Gamma}$$

is an isomorphism. Hence x satisfies Definition 1.9(2). $\qquad\square$

Corollary 5.8. *The subset of saturated points X_s is a Zariski open and dense subspace of X.*

Proof. By Proposition 5.6, X_s is Zariski open in X. It remains to show that it is Zariski dense. Now for any affinoid subdomain $M(S)$ of X, it follows from Proposition 5.7 that $M(S)_{P(k)} \subset X_s$ once k is sufficiently large. Since $M(S)_{P(k)}$ is Zariski dense in $M(S)$, the corollary follows. $\qquad\square$

In the rest of this subsection, we will determine a large class of points $x \in X$ which is contained in the saturated locus X_s (Proposition 5.13). To do this, we need to employ the following flatification result. Let

$$\pi : Y' \to Y$$

be a proper and birational morphism of separated and reduced rigid analytic spaces over E. Here birational means that for some coherent sheaf of ideals H, the complement U of the closed subset $V(H)$, which is defined by H, is Zariski dense in Y, the restriction of π to $\pi^{-1}(U)$ is an isomorphism, and $\pi^{-1}(U)$ is Zariski dense in Y'. Let N be a coherent sheaf of $\mathcal{O}_X$-modules. If H' is the coherent sheaf of ideal defining the closed subset $\pi^{-1}(V(H))$ of X', then the *strict transform N'* of N by π is the quotient of π^*N by its H'^{∞}-torsion. In particular, for any morphism $\pi^*N \to M$ of coherent sheaves over X', if M is torsion-free, then the morphism $\pi^*N \to M$ factors through N'. The following lemma follows from [1, Lemma 3.4.2].

Lemma 5.9. *Let Y be a separated and reduced rigid analytic space over E. If M is a torsion-free coherent sheaf of modules over Y, then there exists a proper and birational morphism $Y' \to Y$ of rigid analytic spaces with Y' reduced such that the strict transform of M by π is a locally free coherent sheaf of modules N over Y'. More precisely, we may choose π to be the blow-up along a nowhere dense Zariski closed subspace of the normalization of Y.*

In the rest of this subsection let V_Y be a locally free coherent $\mathcal{O}_Y$-module of rank d equipped with a continuous $\mathcal{O}_Y$-linear G_K-action. We denote by d_n the rank of $\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)_{\tau}/(t^k)$ as a locally free $\mathcal{O}_Y$-module for any $\tau \in H_K$ (it is independent of τ).

Lemma 5.10. *Let V_Y be a locally free coherent $\mathcal{O}_Y$-module of rank d equipped with a continuous $\mathcal{O}_Y$-linear G_K-action. Suppose for some $n > 0$, the coherent sheaf $\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)/(t^k)$ is well-defined. Now let $\pi : Y' \to Y$ be as in Lemma 5.9, and suppose for some $\tau \in \mathrm{H}_K$, the strict transforms of $(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)/(t^k))_\tau/(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)_\tau/(t^k))_\tau^\Gamma$ and $(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)/(t^k))_\tau^\Gamma$ by π are locally free over $\mathcal{O}_{Y'}$ of ranks c and $d_n - c$ respectively. Then $(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau/(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau^\Gamma$ and $(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau^\Gamma$ are locally free over $\mathcal{O}_{Y'}$ of ranks c and $d_n - c$ respectively as well.*

Proof. We denote by $\mathscr{D}_1$ the strict transform of $(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)/(t^k))_\tau^\Gamma$, and by $\mathscr{D}_2$ the strict transform of $(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)/(t^k))_\tau/(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)/(t^k))_\tau^\Gamma$. Since Y' is normal by Lemma 5.9, it is a disjoint union of irreducible components. By Proposition 2.23, we see both $(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau^\Gamma$ and $(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau/(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau^\Gamma$ are torsion-free on each irreducible component of Y'. Hence the natural maps

$$\pi^*((\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)/(t^k))_\tau^\Gamma) \to (\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau^\Gamma$$

and

$$\pi^*((\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)/(t^k))_\tau/(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)/(t^k))_\tau^\Gamma)$$
$$\to (\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau/(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau^\Gamma$$

factor through $\mathscr{D}_1$ and $\mathscr{D}_2$ respectively. Similarly, since $(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau$ is torsion-free, the natural map

$$\pi^*((\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)/(t^k))_\tau^\Gamma) \to \pi^*(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)/(t^k))_\tau \cong (\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau$$

factors through $\mathscr{D}_1$. To conclude, consider the following commutative diagram

$$
\begin{array}{ccccc}
\mathscr{D}_1 & \longrightarrow & \pi^*(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_Y)/(t^k))_\tau & \longrightarrow & \mathscr{D}_2 \\
\downarrow & & \downarrow{\scriptstyle\simeq} & & \downarrow \\
0 \to (\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau^\Gamma \to & (\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau & \to (\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau/(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau^\Gamma \to 0
\end{array}
$$

where the top sequence satisfies that the second map is surjective and the composite map is zero. By diagram chasing, we see that the right vertical map is surjective, and its kernel is isomorphic to the cokernel of the left vertical map. Since π is an isomorphism on $\pi^{-1}(U)$, the map $\mathscr{D}_1 \to (\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau^\Gamma$ is an isomorphism on $\pi^{-1}(U)$ by Proposition 2.21. It follows that the kernel of

$$\mathscr{D}_2 \to (\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau/(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau^\Gamma$$

is supported on $Y' \setminus \pi^{-1}(U)$, which is a nowhere dense Zariski closed subspace of Y'. Therefore the kernel is zero because $\mathscr{D}_2$ is locally free and Y' is reduced. Hence the right vertical map is an isomorphism. Thus

$$(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau/(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau^\Gamma$$

is locally free of rank $d_n - c$. This implies that $(\mathscr{D}_{\mathrm{dif}}^{+,n}(V_{Y'})/(t^k))_\tau^\Gamma$ is locally free of rank c. $\square$

Lemma 5.11. *Keep notations as above. Suppose that the Sen polynomial for V_Y has no constant term, and that $Y_{fs} = Y$ with respect to the pair (V_Y, α) for some $\alpha \in \mathcal{O}(Y)^\times$. If $\pi : Y' \to Y$ is a proper birational morphism with Y' reduced, the finite slope subspace of Y' with respect to $(\pi^*\alpha, V_{Y'})$ is Y' itself.*

Proof. It is clear that Y' satisfies Definition 1.1(2). Furthermore, since π is birational and Y' is reduced, we also deduce that Y' satisfies Definition 1.1(1). $\qquad\square$

Lemma 5.12. *Keep assumptions as in Lemma 5.11. Moreover, suppose there exists an integer k satisfying*

$$k > \log_{|\pi_K^{-1}|} |\alpha^{-1}|.$$

If $Y' \to Y$ is a proper and birational morphism as in Lemma 5.9 such that for any $\tau \in H_K$, the strict transforms of

$$(\mathscr{D}_{\mathrm{dif}}^{+,fn}(V_Y)/(t^k))_\tau^\Gamma \quad and \quad (\mathscr{D}_{\mathrm{dif}}^{+,fn}(V_Y)/(t^k))_\tau/(\mathscr{D}_{\mathrm{dif}}^{+,fn}(V_Y)/(t^k))_\tau^\Gamma$$

by π are locally free of rank c and $d_n - c$ respectively, then $(\mathscr{D}_{\mathrm{rig}}^\dagger(V_{Y'}))_\sigma^{\varphi^f = \pi^*\alpha, \Gamma = 1}$ is locally free of rank c and

$$\mathscr{D}_{\mathrm{rig}}^\dagger(V_{Y'})_\sigma^{\varphi^f = \pi^*\alpha, \Gamma = 1} \otimes k(y) \to D_{\mathrm{rig}}^\dagger(V_y)_\sigma$$

is injective for any $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$ and $y \in Y'$.

Proof. By Lemma 5.11, $Y'_{fs} = Y'$. Since $k > \log_{|\pi_K^{-1}|} |\alpha^{-1}|$, the natural map

$$(K \otimes_{K_0} \mathscr{D}_{\mathrm{rig}}^\dagger(V_{Y'})_\sigma)^{\varphi^f = \pi^*\alpha, \Gamma = 1} \to \oplus_{\tau \in H_\sigma} (\mathscr{D}_{\mathrm{dif}}^{+,fn}(V_{Y'})/(t^k))_\tau^\Gamma$$

is an isomorphism by Theorem 4.12. By Lemma 5.10, $(\mathscr{D}_{\mathrm{dif}}^{+,fn}(V_{Y'})/(t^k))_\tau^\Gamma$ is locally free of rank c, so $(\mathscr{D}_{\mathrm{rig}}^\dagger(V_{Y'}))_\sigma^{\varphi^f = \pi^*\alpha, \Gamma = 1}$ as well. Furthermore, since $(\mathscr{D}_{\mathrm{dif}}^{+,fn}(V_{Y'})/(t^k))_\tau/(\mathscr{D}_{\mathrm{dif}}^{+,fn}(V_{Y'})/(t^k))_\tau^\Gamma$ is also locally free by Lemma 5.10, we deduce that

$$(\mathscr{D}_{\mathrm{dif}}^{+,fn}(V_{Y'})/(t^k))_\tau^\Gamma \otimes k(y) \to (\mathscr{D}_{\mathrm{dif}}^{+,fn}(V_{Y'})/(t^k))_\tau \otimes k(y) \cong (D_{\mathrm{dif}}^{+,fn}(V_y)/(t^k))_\tau$$

is injective for any $y \in Y'$; the isomorphism follows from the fact that the functor $D_{\mathrm{dif}}^{+,n}$ is compatible with base change (Proposition 2.14). This implies that the map

$$(K \otimes_{K_0} \mathscr{D}_{\mathrm{rig}}^\dagger(V_{Y'}))_\sigma^{\varphi^f = \pi^*\alpha, \Gamma = 1} \otimes k(y) \to \oplus_{\tau \in H_\sigma} (\mathscr{D}_{\mathrm{dif}}^{+,fn}(V_y)/(t^k))_\tau$$

is injective. Note that this map factors through $(K \otimes_{K_0} D_{\mathrm{rig}}^\dagger(V_y)_\sigma)^{\varphi^f = \alpha(\pi(y)), \Gamma = 1}$. The lemma follows. $\qquad\square$

Proposition 5.13. *For any $x \in X$ and $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, if*

$$\dim_{k(x)} D^{\dagger}_{\mathrm{rig}}(V^{ss}_x)^{\varphi^f = F(x), \Gamma = 1}_{\sigma} = 1,$$

then $\mathscr{D}^{\dagger}_{\mathrm{rig}}(V_X)^{\varphi^f = F, \Gamma = 1}_{\sigma}$ is locally free of rank 1 around x, and

$$\mathscr{D}^{\dagger}_{\mathrm{rig}}(V_X)^{\varphi^f = F, \Gamma = 1}_{\sigma} \otimes k(x) \to D^{\dagger}_{\mathrm{rig}}(V_x)^{\varphi^f = F(x), \Gamma = 1}_{\sigma}$$

is an isomorphism

Proof. We may suppose $X = M(S)$ is an affinoid space. Let $\widetilde{X}$ be the normalization of X. It follows that $\widetilde{X}$ is the disjoint union of finitely many irreducible components. By Proposition 2.23, both

$$(\mathscr{D}^{+,n}_{\mathrm{dif}}(V_{\widetilde{X}})/(t^k))^{\Gamma}_{\tau} \text{ and } (\mathscr{D}^{+,n}_{\mathrm{dif}}(V_{\widetilde{X}})/(t^k))_{\tau}/(\mathscr{D}^{+,fn}_{\mathrm{dif}}(V_{\widetilde{X}})/(t^k))^{\Gamma}_{\tau}$$

are torsion-free on each irreducible component of $\widetilde{X}$. Using Lemmas 5.9, 5.10 and 5.12, there exists a proper birational map $\widetilde{\pi} : X' \to \widetilde{X}$ such that

$$(\mathscr{D}^{\dagger}_{\mathrm{rig}}(V_{X'}))^{\varphi^f = \pi^* F, \Gamma = 1}_{\sigma}$$

is locally free of rank 1, and

$$(\mathscr{D}^{\dagger}_{\mathrm{rig}}(V_{X'}))^{\varphi^f = \pi^* F, \Gamma = 1}_{\sigma} \otimes k(x') \to D^{\dagger}_{\mathrm{rig}}(V_{x'})_{\sigma}$$

is injective for any $x' \in X'$. In particular, the map is nonzero. Therefore for any ideal I of cofinite length of $\mathcal{O}_{X',x'}$, the composite

$$(\mathscr{D}^{\dagger}_{\mathrm{rig}}(V_{X'}))^{\varphi^f = \pi^* F, \Gamma = 1}_{\sigma} \otimes_{\mathcal{O}_{X'}} (\mathcal{O}_{X',x'}/I)$$
$$\to (D^{\dagger}_{\mathrm{rig}}(V_{X'} \otimes_{\mathcal{O}_{X'}} (\mathcal{O}_{X',x'}/I)))^{\varphi^f = \pi^* F, \Gamma = 1}_{\sigma}$$
$$\to (D^{\dagger}_{\mathrm{rig}}(V_{x'}))^{\varphi^f = (\pi^* F)(x'), \Gamma = 1}_{\sigma}$$

is nonzero.

Now let π be the composite $X' \to \widetilde{X} \to X$, which is also birational. If

$$x' \in \pi^{-1}(x),$$

applying [1, Lemma 3.3.9] to the functor $D^+_{\mathrm{crys}}(\cdot)^{\varphi^f = \pi^* F}_{\sigma}$, we deduce

$$D^+_{\mathrm{crys}}(V_{X'} \otimes_{\mathcal{O}_{X'}} (\mathcal{O}_{X',x'}/I))^{\varphi^f = \pi^* F}_{\sigma}$$

is free of rank 1 over $\mathcal{O}_{X',x'}/I$. Therefore, by [1, Proposition 3.2.3], for all ideals I of cofinite length of $\mathcal{O}_{X,x}$, $D^+_{\mathrm{crys}}(V_X \otimes_{\mathcal{O}_X} (\mathcal{O}_{X,x}/I))^{\varphi^f = F}_{\sigma}$ is free of rank 1 over $\mathcal{O}_{X,x}/I$.

Furthermore, we claim that if $I' \supset I$ is another ideal of $\mathcal{O}_x$, the natural map

$$D_{\text{crys}}^+(V_X \otimes_{\mathcal{O}_X} (\mathcal{O}_{X,x}/I))_\sigma^{\varphi^f=F} \to D_{\text{crys}}^+(V_X \otimes_{\mathcal{O}_X} (\mathcal{O}_{X,x}/I'))_\sigma^{\varphi^f=F} \qquad (5.4)$$

is surjective. In fact, since D_{crys}^+ is left exact, we first have the following exact sequence

$$0 \to D_{\text{crys}}^+(V_X \otimes_{\mathcal{O}_X} (I'/I))_\sigma^{\varphi^f=F} \to D_{\text{crys}}^+(V_X \otimes_{\mathcal{O}_X} (\mathcal{O}_{X,x}/I))_\sigma^{\varphi^f=F}$$
$$\to D_{\text{crys}}^+(V_X \otimes_{\mathcal{O}_X} (\mathcal{O}_{X,x}/I'))_\sigma^{\varphi^f=F}.$$

This implies that

$$l(D_{\text{crys}}^+(V_X \otimes_{\mathcal{O}_X} (I'/I))_\sigma^{\varphi^f=F})$$
$$\geq l(D_{\text{crys}}^+(V_X \otimes_{\mathcal{O}_X} (\mathcal{O}_{X,x}/I))_\sigma^{\varphi^f=F}) - l(D_{\text{crys}}^+(V_X \otimes_{\mathcal{O}_X} (\mathcal{O}_{X,x}/I'))_\sigma^{\varphi^f=F})$$
$$= l(\mathcal{O}_{X,x}/I) - l(\mathcal{O}_{X,x}/I')$$
$$= l(I'/I).$$

On the other hand, we deduce from the assumption that

$$l(D_{\text{crys}}^+(V_X \otimes_{\mathcal{O}_X} (I'/I))_\sigma^{\varphi^f=F}) \leq l(I'/I).$$

This forces $l(D_{\text{crys}}^+(V_X \otimes_{\mathcal{O}_X} (I'/I))_\sigma^{\varphi^f=F}) = l(I'/I)$. Hence (5.4) is surjective.

Now choose a positive integer $k > \log_{|\pi_K^{-1}|} |F^{-1}|$. By Theorem 5.4, the map

$$(K \otimes_{K_0} D_{\text{rig}}^\dagger(V_S)_\sigma)^{\varphi^f=F,\Gamma=1} \to \oplus_{\tau \in \text{H}_\sigma} (D_{\text{dif}}^{+,n}(V_S)/(t^k))_\tau^\Gamma$$

is an isomorphism. Since $\widehat{\mathcal{O}}_{X,x}$ is flat over S, we deduce from Lemma 2.19 that

$$(D_{\text{dif}}^{+,fn}(V_S)/(t^k))_\tau^\Gamma \otimes_S \widehat{\mathcal{O}}_{X,x} \cong ((D_{\text{dif}}^{+,fn}(V_S)/(t^k))_\tau \otimes_S \widehat{\mathcal{O}}_{X,x})^\Gamma.$$

Since $(D_{\text{dif}}^{+,fn}(V_S)/(t^k))_\tau$ is finite locally free over S, we get

$$(D_{\text{dif}}^{+,fn}(V_S)/(t^k))_\tau \otimes_S \widehat{\mathcal{O}}_{X,x} \cong \varprojlim_l (D_{\text{dif}}^{+,fn}(V_S)/(t^k))_\tau \otimes_S S/\mathfrak{m}_x^l)$$
$$\cong \varprojlim_l (D_{\text{dif}}^{+,fn}(V_S/\mathfrak{m}_x^l V_S)/(t^k))_\tau,$$

where the last isomorphism follows from the base change property of the functor $D_{\text{dif}}^{+,fn}$. Hence

$$((D_{\text{dif}}^{+,fn}(V_S)/(t^k))_\tau \otimes_S \widehat{\mathcal{O}}_{X,x})^\Gamma \cong \varprojlim_l ((D_{\text{dif}}^{+,fn}(V_S)/(t^k))_\tau \otimes_S S/\mathfrak{m}_x^l)^\Gamma$$
$$\cong \varprojlim_l (D_{\text{dif}}^{+,fn}(V_S/\mathfrak{m}_x^l V_S)/(t^k))_\tau^\Gamma.$$

Now consider the following commutative diagram

$$(K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S)_{\sigma})^{\varphi^f = F, \Gamma = 1} \otimes_S \widehat{\mathcal{O}}_{X,x} \longrightarrow \oplus_{\tau \in \mathrm{H}_{\sigma}} (\mathrm{D}_{\mathrm{dif}}^{+,fn}(V_S)_{\tau}/(t^k))^{\Gamma} \otimes_S \widehat{\mathcal{O}}_{X,x}$$

$$\varprojlim_l (K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S/\mathfrak{m}_x^l V_S)_{\sigma})^{\varphi^f = F, \Gamma = 1} \longrightarrow \varprojlim_l \oplus_{\tau \in \mathrm{H}_{\sigma}} (\mathrm{D}_{\mathrm{dif}}^{+,fn}(V_S/\mathfrak{m}_x^l V_S)_{\tau}/(t^k))^{\Gamma}.$$

By the previous paragraph we see that $\varprojlim_l \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S/\mathfrak{m}_x^l V_S)_{\sigma}^{\varphi^f = F, \Gamma = 1}$ is a free $\widehat{\mathcal{O}}_{X,x}$-module of rank 1. Since both the top horizontal and right vertical maps are isomorphisms, we deduce that the left vertical map embeds

$$(K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S)_{\sigma})^{\varphi^f = F, \Gamma = 1} \otimes_S \widehat{\mathcal{O}}_{X,x}$$

as a direct summand of

$$\varprojlim_l (K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S/\mathfrak{m}_x^l V_S)_{\sigma})^{\varphi^f = F, \Gamma = 1} = K \otimes_{K_0} \varprojlim_l \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S/\mathfrak{m}_x^l V_S)_{\sigma}^{\varphi^f = F, \Gamma = 1}.$$

It follows that the map

$$(K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S))_{\sigma}^{\varphi^f = F, \Gamma = 1} \otimes_S k(x) \to (K \otimes_{K_0} \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_x))_{\sigma}^{\varphi^f = F(x), \Gamma = 1}$$

is injective. On the other hand, we have

$$\dim(\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S)_{\sigma}^{\varphi^f = F, \Gamma = 1} \otimes_S k(x)) \geq 1$$

by Proposition 5.5(3). Thus the left hand side is at least $[K : \mathbb{Q}_p]/f$-dimensional whereas the right hand side is exactly $[K : \mathbb{Q}_p]/f$-dimensional. Hence

$$\dim_{k(x)}(\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S)_{\sigma}^{\varphi^f = F, \Gamma = 1} \otimes_S k(x)) = 1$$

and

$$\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S)_{\sigma}^{\varphi^f = F, \Gamma = 1} \otimes_S k(x) \to \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_x)_{\sigma}^{\varphi^f = F(x), \Gamma = 1}$$

is an isomorphism. $\qquad\square$

5.2. Vector bundles and (φ, Γ)-modules.

In this subsection, we will recall some basic notions and properties of the theory of families (φ, Γ)-modules. Recall that we denote by K_0' the maximal unramified extension of $\mathbb{Q}_p$ contained in K_∞. Let S be an affinoid algebra over $\mathbb{Q}_p$.

Definition 5.14. Let I be a subinterval of $(0, \infty)$. By a *vector bundle* over $\mathcal{R}^I_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ of rank d we mean a locally free coherent sheaf M^I_S of rank d over the product of the annulus $v_p(T) \in I$ within the affine T-line over K'_0 with $M(S)$ in the category of rigid analytic spaces over $\mathbb{Q}_p$. We call M^I_S *free* if it is freely generated by its global sections. By a *vector bundle* M_S over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ we mean an object in the direct limit as $r \to 0$ of the categories of vector bundles over $\mathcal{R}^r_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$.

For a subinterval I' of I, denote by $M^{I'}_S$ the base change of M^I_S to $\mathcal{R}^{I'}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$. If $S \to R$ is a map of affinoid algebras over $\mathbb{Q}_p$, we set M^I_R and M_R as the base changes of M^I_S and M_S to $\mathcal{R}^I_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} R$ and $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} R$ respectively. For any $x \in M(S)$, we denote $M^I_{k(x)}$ and $M_{k(x)}$ by M^I_x and M_x respectively instead.

Remark 5.15. By Lemma 3.7, $M^r_x = M^r_S \otimes_S k(x)$. Hence the map $M^r_S \to M^r_x$ is surjective.

Remark 5.16. A locally free $\mathcal{R}^I_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$-module of rank d naturally gives rise to a vector bundle of rank d over $\mathcal{R}^I_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$. The converse is also true when I is a closed interval.

We need the following result, which is originally due to Lütkebohmert [28], in §5.3.

Lemma 5.17. *Let M^I_S be a vector bundle over $\mathcal{R}^I_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$. If I is closed, then there exists a finite covering of $M(S)$ by affinoid subdomains $M(S_1), \ldots, M(S_i)$ such that $M^I_{S_1}, \ldots, M^I_{S_i}$ are all free.*

Recall that there exists an isomorphism $\mathbf{B}^\dagger_{\mathrm{rig}, K} \cong \mathcal{R}_{K'_0}$, which identifies $\mathbf{B}^{\dagger, \rho(r)}_{\mathrm{rig}, K}$ with $\mathcal{R}^r_{K'_0}$ for all sufficiently small r. We henceforth identify $\mathbf{B}^\dagger_{\mathrm{rig}, K} \widehat{\otimes}_{\mathbb{Q}_p} S$ with $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$, and equip the latter with the induced φ- and Γ-actions.

Definition 5.18. By a *(φ, Γ)-module* over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ of rank d we mean a vector bundle D_S over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ of rank d equipped with commuting semilinear φ- and Γ-actions such that the induced map $\varphi^* D_S \to D_S$ is an isomorphism as vector bundles over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$. We say D_S *free* if the underlying vector bundle is free. The morphisms of (φ, Γ)-modules over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ are morphisms of the underlying vector bundles which respect φ- and Γ-actions.

Definition 5.19. Let D_S be a (φ, Γ)-module over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$. It is clear from Definition 5.18 that for r sufficiently small, D_S is represented by a vector bundle $D^r_S \subset D_S$ over $\mathcal{R}^r_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ such that φ maps D^r_S to $D^{r/p}_S$, and the induced map $\varphi^*(D^r_S) \to D^{r/p}_S$ is an isomorphism as vector bundles over $\mathcal{R}^{r/p}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$. We call such D^r_S *representative vector bundles* of D_S.

Remark 5.20. Our definition of (φ, Γ)-modules over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ is the same as the notion of *families of (φ, Γ)-modules* over $\mathcal{R}_{S_{K'}}$ defined in [22].

Remark 5.21. If V_S is a locally free S-linear representation of rank d of G_K, $D_{\mathrm{rig}}^{\dagger}(V_S)$ is naturally a (φ, Γ)-module of rank d over $\mathcal{R}_{K_0'} \widehat{\otimes}_{\mathbb{Q}_p} S$ with representative vector bundles $D_{\mathrm{rig}}^{\dagger,\rho(r)}(V_S)$ for r sufficiently small.

Remark 5.22. If S is a finite extension of $\mathbb{Q}_p$, then D_S is free over $\mathcal{R}_{K_0'} \widehat{\otimes}_{\mathbb{Q}_p} S = \mathcal{R}_{K_0'} \otimes_{\mathbb{Q}_p} S$ by the Bézout property of $\mathcal{R}_{K_0'} \otimes_{\mathbb{Q}_p} S$. Thus our definition of (φ, Γ)-modules is compatible with the definition of classical (φ, Γ)-modules.

Remark 5.23. In fact, one can also define a (φ, Γ)-module over $\mathcal{R}_{K_0'} \widehat{\otimes}_{\mathbb{Q}_p} S$ of rank d to be a finite presented projective module over $\mathcal{R}_{K_0'} \widehat{\otimes}_{\mathbb{Q}_p} S$ of rank d equipped with commuting semilinear φ- and Γ-actions such the induce map $\varphi^*(D_S) \to D_S$ is an isomorphism. The equivalence between this definition and ours are proved in [24] and [6] independently.

Lemma 5.24. *Let L be a finite extension of $\mathbb{Q}_p$, and put $L' = L \otimes_{\mathbb{Q}_p} K_0'$. Let D be a (φ, Γ)-module over $\mathcal{R}_{K_0'} \otimes_{\mathbb{Q}_p} L = \mathcal{R}_{L'}$ of rank n, and let E be a (φ, Γ)-submodule of D of rank m. Then there exists an $r_0 > 0$ such that if D^r and E^r are representative vector bundles of D and E for some $r \le r_0$, then $E^r \subset D^r$. As a consequence, D has at most one representative vector bundle over $\mathcal{R}_{L'}^r$ when r is sufficiently small.*

Proof. Fix some $r_0 > 0$ such that for any $a \in \mathcal{R}_{K_0'}$, if $\varphi(a) \in \mathcal{R}_{K_0'}^r$ for some $0 < r \le r_0$, then $a \in \mathcal{R}_{K'}^{pr}$. Now let $\mathrm{d} = (d_1, \ldots, d_n)$ and $\mathrm{e} = (e_1, \ldots, e_m)$ be $\mathcal{R}_{L'}^r$-bases of D^r and E^r respectively. Since D^r and E^r are representative vector bundles, there exist invertible matrices A and B defined over $\mathcal{R}_{L'}^{r/p}$ such that $\varphi(\mathrm{d}) = \mathrm{d}A$ and $\varphi(\mathrm{e}) = \mathrm{e}B$. Write $\mathrm{e} = \mathrm{d}C$ for some $n \times m$ matrix C defined over $\mathcal{R}_{L'}$. It follows

$$\mathrm{d}CB = \mathrm{e}B = \varphi(\mathrm{e}) = \varphi(\mathrm{d})\varphi(C) = \mathrm{d}A\varphi(C),$$

yielding $CB = A\varphi(C)$. Hence $\varphi(C) = A^{-1}CB$. Now suppose C is defined over $\mathcal{R}_{L'}^s$ for some $s > 0$. If $s < r/p$, then $\varphi(C) = A^{-1}CB$ is over $\mathcal{R}_{L'}^s$, yielding that C is defined over $\mathcal{R}_{L'}^{ps}$. Iterating this argument, we conclude that C is defined over $\mathcal{R}_{L'}^{r/p}$. Thus $\varphi(C)$ is defined over $\mathcal{R}_{L'}^{r/p}$, yielding C is defined over $\mathcal{R}_{L'}^r$. This implies $E^r \subset D^r$. $\square$

Lemma 5.25. *Keep notations as in the Lemma 5.24. Then E is saturated in D if and only if E^r is saturated in D^r. Furthermore, in this case, we have*

$$E^r = D^r \cap E,$$

and D^r/E^r is the representative vector bundle of D/E over $\mathcal{R}_{L'}^r$.

Proof. It is obvious that if E^r is saturated in D^r, then E is saturated in D. Now suppose E is saturated in D. First note that $D^r/(E \cap D^r)$ is a submodule of D/E.

Hence $D^r/(E \cap D^r)$ is finitely generated and torsion-free over $\mathcal{R}^r_{L'}$. This yields that it is finite free over $\mathcal{R}^r_{L'}$ by the Bézout property of $\mathcal{R}^r_{L'}$. Furthermore, since it generates D/E over $\mathcal{R}_{L'}$, we deduce

$$\operatorname{rank}_{\mathcal{R}^r_{L'}}(D^r/(E \cap D^r)) \geq \operatorname{rank}_{\mathcal{R}_{L'}}(D/E) = d - s.$$

On the other hand, $E \cap D^r$ is a closed $\mathcal{R}^r_{L'}$-submodule of D^r. Hence it is also finite free over $\mathcal{R}^r_{L'}$. Since $E^r \subseteq E \cap D^r$, we deduce

$$\operatorname{rank}_{\mathcal{R}^r_{L'}}(E \cap D^r) \geq \operatorname{rank}_{\mathcal{R}^r_{L'}} E^r = s.$$

Since

$$\operatorname{rank}(E \cap D^r) + \operatorname{rank}(D^r/(E \cap D^r)) = \operatorname{rank} D^r = d,$$

we deduce

$$\operatorname{rank}(E \cap D^r) = s \quad \text{and} \quad \operatorname{rank}(D^r/(E \cap D^r)) = d - s.$$

We claim that $E \cap D^r$ and $D^r/(E \cap D^r)$ are representative vector bundles of E and D/E respectively. First note that the natural map

$$(D^r/(E \cap D^r)) \otimes_{\mathcal{R}^r_{L'}} \mathcal{R}_{L'} \to D/E$$

is an isomorphism because it is surjective, and both sides are finite free over $\mathcal{R}_{L'}$ of the same rank. It follows that

$$(E \cap D^r) \otimes_{\mathcal{R}^r_{L'}} \mathcal{R}_{L'} \to E$$

is also an isomorphism. Now consider the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \varphi^*(E \cap D^r) & \longrightarrow & \varphi^*(D^r) & \longrightarrow & \varphi^*(D^r/(E \cap D^r)) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (E \cap D^r) \otimes_{\mathcal{R}^r_{L'}} \mathcal{R}^{r/p}_{L'} & \longrightarrow & D^{r/p} & \longrightarrow & D^{r/p}/((E \cap D^r) \otimes_{\mathcal{R}^r_{L'}} \mathcal{R}^{r/p}_{L'}) & \longrightarrow & 0.
\end{array}
$$

The middle vertical map is an isomorphism as D^r is a representative vector bundle. Thus the right vertical map is surjective. Hence it is an isomorphism because both the source and target are finite free of the same rank over $\mathcal{R}^{r/p}_{L'}$. This yields that the left vertical map is also an isomorphism. The claim now follows, and we deduce the lemma from Lemma 5.24. $\square$

Proposition 5.26. *Keep notations as above. Let D_S be a (φ, Γ)-module over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ of rank d, and let E_S be a (φ, Γ)-submodule of D_S of rank s. Suppose $E^r_S \subset D^r_S$ are representative vector bundles of E_S and D_S respectively. If E_x is a saturated (φ, Γ)-submodule of D_x for every $x \in M(S)$, D^r_S/E^r_S is a vector bundle over $\mathcal{R}^r_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ of rank $d - s$. As a consequence, D_S/E_S is a (φ, Γ)-module over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ of rank $d - s$.*

Proof. It suffices to show that E^r_x is saturated in D^r_x for every $x \in M(S)$. The latter follows from Lemma 5.25. $\qquad\square$

The following lemma has been used in the proof of Proposition 5.6.

Lemma 5.27. *Keep notations as above. Suppose L is a finite extension of E. Let V be a L-linear representation of G_K of dimension d. Let D_1 be a rank 1 (φ, Γ)-submodule over $\mathcal{R}_{L'}$ of $D = D^{\dagger}_{\mathrm{rig}}(V)$. Fix an integer n so that the degree of field extensions $[K_m : \mathbb{Q}_p(\epsilon_m)]$ are constant for all $m \geq n$, and $D^{\rho(r_n)}_1 \subset D^{\rho(r_n)}$. Then D_1 is saturated in D if and only if $D^{\rho(r_n)}_1$ has nonzero image in $D^n_{\mathrm{Sen}}(V)_\tau$ via the composite*

$$\iota_n : D^{\rho(r_n)} = D^{\dagger,r_n}_{\mathrm{rig}}(V) \to D^{+,n}_{\mathrm{dif}}(V) \to D^n_{\mathrm{Sen}}(V) \to D^n_{\mathrm{Sen}}(V)_\tau$$

for any $\tau \in H_K$.

Proof. The "only if" part is obvious. It remains to prove the "if" part. To do this, we apply induction for (φ, Γ)-modules defined in [26]. Using the set up of [26], $\mathrm{Ind}^{\Gamma_{\mathbb{Q}_p}}_{\Gamma_K} D_1$ and $\mathrm{Ind}^{\Gamma_{\mathbb{Q}_p}}_{\Gamma_K} D$ are $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules over $\mathcal{R}_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L = \mathcal{R}_L$ of ranks $[K : \mathbb{Q}_p]$ and $d[K : \mathbb{Q}_p]$ respectively. Furthermore,

$$\mathrm{Ind}^{\Gamma_{\mathbb{Q}_p}}_{\Gamma_K} D = D^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}(\mathrm{Ind}^{G_K}_{G_{\mathbb{Q}_p}} V)$$

since inductions for p-adic representations are compatible with inductions for the associated (φ, Γ)-modules [26, Proposition 2.1]. Then it suffices to show that $\mathrm{Ind}^{\Gamma_{\mathbb{Q}_p}}_{\Gamma_K} D_1$ is a saturated $(\varphi, \Gamma_{\mathbb{Q}_p})$-submodule of rank $h = [K : \mathbb{Q}_p]$ of $\mathrm{Ind}^{\Gamma_{\mathbb{Q}_p}}_{\Gamma_K} D$.

Suppose the contrary is true. Using [26, Proposition 3.1], we first deduce that as an $L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\epsilon_n)$-module, the image of $\mathrm{Ind}^{\Gamma_{\mathbb{Q}_p}}_{\Gamma_K} D^{\rho(r_n)}_1$ in $D^n_{\mathrm{Sen}}(\mathrm{Ind}^{\Gamma_{\mathbb{Q}_p}}_{\Gamma_K} V)$ can be generated by $h - 1$ elements. This implies that the image has L-dimension $\leq (h-1)[\mathbb{Q}_p(\epsilon_n) : \mathbb{Q}_p]$. On the other hand, since Γ_K acts transitively on the set of components $(L \otimes_K K_n)_\tau$, the image of $D^{\rho(r_n)}_1$ in $D^n_{\mathrm{Sen}}(V)_\tau$ has L-dimension at least $[K_n : K]$. Now by the assumption on n, we have

$$D^n_{\mathrm{Sen}}(\mathrm{Ind}^{\Gamma_{\mathbb{Q}_p}}_{\Gamma_K} V) = \mathrm{Ind}^{\Gamma_{\mathbb{Q}_p}}_{\Gamma_K}(D^n_{\mathrm{Sen}}(V)) = \oplus_{\tau \in H_K} \mathrm{Ind}^{\Gamma_{\mathbb{Q}_p}}_{\Gamma_K}(D^n_{\mathrm{Sen}}(V)_\tau).$$

It follows that the image of $\mathrm{Ind}^{\Gamma_{\mathbb{Q}_p}}_{\Gamma_K} D^{\rho(r_n)}_1$ in $D^n_{\mathrm{Sen}}(\mathrm{Ind}^{\Gamma_{\mathbb{Q}_p}}_{\Gamma_K} V)$ has L-dimension at least

$$h[\Gamma_{\mathbb{Q}_p} : \Gamma_K][K_n : K] = h(h/[K_n : \mathbb{Q}_p(\epsilon_n)])[K_n : K]$$
$$= h[K_n : \mathbb{Q}_p]/[K_n : \mathbb{Q}_p(\epsilon_n)] = h[\mathbb{Q}_p(\epsilon_n) : \mathbb{Q}_p].$$

This yields a contradiction! $\qquad\square$

Remark 5.28. In the case when $S = L$ is a finite extension of $\mathbb{Q}_p$, for any $\delta \in \widehat{\mathscr{T}}(S)$, Nakamura constructs a rank 1 B-pair $W(\delta)$ [30]. A short computation shows that $\mathcal{R}_L(\delta)$ is isomorphic to the (φ, Γ)-module corresponding to $W(\delta)$. Therefore, for an L-linear representation of G_K, being trianguline with parameters $(\delta_i)_{1 \leq i \leq d}$ in the sense of Definition 1.4 is the same as being split triabguline in the sense of Nakamura with the same set of parameters.

5.3. Refined families. The main goal of this subsection is prove the main result of this paper. That is, a family of refined p-adic representations of G_K admits a global triangulation on a Zariski open and dense subspace of the base that contains all regular non-critical points. In what follows, we first give the definition of regular non-critical refined p-adic representations.

Definition 5.29. Let L be a finite extension of E, and let V be a d-dimensional crystalline L-linear representation of G_K such that φ^f acting on $D_{\mathrm{crys}}(V)$ has all its eigenvalues in $L^{\times}$.

(1) By a *refinement* of V we mean a φ-stable $K_0 \otimes_{\mathbb{Q}_p} L$-filtration $\mathcal{F} = (\mathcal{F}_i)_{1 \leq i \leq d}$ of $D_{\mathrm{crys}}(V)$:

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \cdots \subsetneq \mathcal{F}_d = D_{\mathrm{crys}}(V).$$

In particular, $\dim \mathcal{F}_i = i$.

(2) For $\tau \in \mathrm{H}_K$, suppose the Hodge–Tate weights of $D_{\mathrm{dR}}(V)_\tau$ are

$$k_{1,\tau} > k_{2,\tau} \cdots > k_{d,\tau}.$$

We say the refinement $\mathcal{F}$ is *τ-non-critical* if

$$D_{\mathrm{dR}}(V)_\tau = (K \otimes_{K_0} \mathcal{F}_i)_\tau \oplus \mathrm{Fil}^{k_i+1,\tau}(D_{\mathrm{dR}}(V)_\tau) \qquad (5.5)$$

for all $1 \leq i \leq d$. The refinement $\mathcal{F}$ is said to be *non-critical* if it is τ-non-critical for every $\tau \in \mathrm{H}_K$.

(3) We denote by φ_i the eigenvalue of φ^f on $\mathcal{F}_i/\mathcal{F}_{i-1}$. We say the refinement $\mathcal{F}$ is *regular* if for any $1 \leq i \leq d$, $\varphi_1 \cdots \varphi_i$ is an eigenvalue of φ^f on $D_{\mathrm{crys}}(\wedge^i V)$ of multiplicity one.

The refinement $\mathcal{F}$ gives rise to an ordering $(\varphi_1, \ldots, \varphi_d)$ of the φ^f-eigenvalues on $D_{\mathrm{crys}}(V)$. If all these eigenvalues are distinct, any ordering of them uniquely gives rise to a refinement. For any $\tau \in \mathrm{H}_K$, the refinement $\mathcal{F}$ also gives rise to an ordering $(s_{1,\tau}, \ldots, s_{d,\tau})$ of $\{k_{1,\tau}, \ldots, k_{d,\tau}\}$, defined by the property that the jumps of the Hodge filtration of $D_{\mathrm{dR}}(V)_\tau$ induced on $(K \otimes_{K_0} \mathcal{F}_i)_\tau$ are $(s_{1,\tau}, \ldots, s_{i,\tau})$. It is straightforward to see that $\mathcal{F}$ is τ-non-critical if and only if the associated ordering of the Hodge–Tate weights is $(k_{1,\tau}, \ldots, k_{d,\tau})$.

From now on, let X be a reduced rigid analytic space over E, and let E be a family of refined p-adic representations of G_K of dimension d over X as in §1.3. In the following, we retain the notations in §1.3 and §1.4.

Remark 5.30. Recall that in §1.3 we define $\alpha_i = \prod_{j=1}^{i} F_j$ and $\eta_i = \prod_{j=1}^{i} \chi_j$ for $1 \leq i \leq d$. If V_X is a refined family of rank d, then for each $1 \leq i \leq d$, the i-th exterior product $\wedge^i V_X$ is a weakly refined family with $F = \alpha_i$, the generalized Hodge–Tate weights

$$\left\{ \kappa_I = \sum_{j \in I} \kappa_j \right\}_{|I|=i},$$

the biggest Hodge–Tate weight $\kappa_1 + \cdots + \kappa_i$ and the same Zariski dense subset Z. Hence $(\wedge^i V_X)(\eta_i^{-1})$ is a weakly refined family with generalized Hodge–Tate weights $\{\kappa_I - \kappa_{\{1,\ldots,i\}}\}_{|I|=i}$ and $F = \alpha_i$. In particular, its biggest Hodge–Tate weight is 0.

By Remark 5.30, we may apply Theorem 5.4 to $(\wedge^i V_X)(\eta_i^{-1})$ to get the following:

Proposition 5.31. *For each* $1 \leq i \leq d$, *the presheaf* $\mathscr{D}_{\mathrm{rig}}^{\dagger}((\wedge^i V_X)(\eta_i^{-1}))^{\varphi^f = \alpha_i, \Gamma = 1}$ *is a coherent sheaf on* X.

For each $1 \leq i \leq d-1$, let $T Q_i(T)$ be the Sen polynomial for $(\wedge^i V_X)(\eta_i^{-1})$, and let

$$P_i(k) = \prod_{j=0}^{k-1} Q_i(-j)$$

for $k \geq 1$. The following proposition follows immediately from Proposition 5.7.

Proposition 5.32. *For* $x \in X$, *if there exist positive integers* $k_i > v_K(\alpha_i(x))$ *for each* $1 \leq i \leq d-1$ *satisfying*

$$(P_1(k_1) \cdots P_{d-1}(k_{d-1}))(x) \neq 0, \tag{5.6}$$

then $x \in X_s$ *and* $\dim \mathrm{D}_{\mathrm{rig}}^{\dagger}((\wedge^i V_x)(\eta_i(x)^{-1}))_{\sigma}^{\varphi^f = \alpha_i(x), \Gamma = 1}$ *for each* $1 \leq i \leq d$ *and* $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$.

Proposition 5.33. *For* $x \in X$, $1 \leq i \leq d$ *and* $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, *if*

$$\dim_{k(x)} \mathrm{D}_{\mathrm{rig}}^{\dagger}((\wedge^i V_x^{\mathrm{ss}})(\eta_i(x)^{-1}))_{\sigma}^{\varphi^f = \alpha_i(x), \Gamma = 1} = 1,$$

the coherent sheaf $\mathscr{D}_{\mathrm{rig}}^{\dagger}((\wedge^i V_X)(\eta_i^{-1}))_{\sigma}^{\varphi^f = \alpha_i, \Gamma = 1}$ *is locally free of rank 1 around* x, *and*

$$\mathscr{D}_{\mathrm{rig}}^{\dagger}((\wedge^i V_X)(\eta_i^{-1}))_{\sigma}^{\varphi^f = \alpha_i, \Gamma = 1} \otimes k(x) \to \mathrm{D}_{\mathrm{rig}}^{\dagger}((\wedge^i V_x)(\eta_i(x)^{-1})))_{\sigma}^{\varphi^f = \alpha_i(x), \Gamma = 1}$$

is an isomorphism.

Proof. We conclude the proposition by applying Proposition 5.13 to the weakly refined family $(\wedge^i V_X)(\eta_i^{-1})$. $\square$

As already noted in §1.4, the saturated locus X_s of V_X is a Zariski open and dense subspace of X. Therefore it reduces to show that V_{X_s} admits a global triangulation on X_s and X_s contains all regular non-critical points. To this end, we will show that the triangulation locus of V_{X_s} forms a reduced Zariski closed subspace of X_s. The upshot is to note that for a sequence of crystalline periods of the successive exterior products of V_x obtained by the previous results, the condition that it gives rise to a triangulation of V_x is purely algebraic. To make this statement precise, we introduce the following notions.

Definition 5.34. Let A be a commutative ring with identity, and let M be a free A-module of rank d.

(1) We call a free A-submodule $N \subseteq M$ of rank c *cofree* if M/N is a free A-module of rank $d - c$. We call $m \in M$ *cofree* if Am is cofree.

(2) Let $m \in M$ be cofree, and let $n \in \wedge^i M$ for some $1 \le i \le d$. Suppose

$$m \wedge n = 0$$

in $\wedge^{i+1} M$. Then there exists a unique $\bar{n} \in \wedge^{i-1}(M/Am)$ such that the wedge product of any lift of $\bar{n}$ in $\wedge^{i-1} M$ with m is equal to n; we call $\bar{n}$ the *quotient* of n by m. Let N be a free rank 1 A-submodule of M, and let P be a free rank 1 A-submodule of $\wedge^i M$. If $N \wedge P = 0$ in $\wedge^{i+1} M$, we define the quotient of P by N to be the A-submodule of $\wedge^{i-1} M$ generated by the quotient of any generator of P by any generator of N.

(3) For each $1 \le i \le d$, let N_i be a free rank 1 A-submodule of $\wedge^i M$. We say the sequence $N_1, \ldots, N_d$ forms a *chain* in M if there exists an A-basis $e_1, \ldots, e_d$ of M such that
$$N_i = Ae_1 \wedge \cdots \wedge e_i$$
for all $1 \le i \le d$. In this case, the filtration

$$\mathrm{Fil}_i(M) = \text{Span of } \{e_j\}_{0 \le j \le i}, \quad 1 \le i \le d - 1,$$

which is independent of the choice of the basis $\{e_1, \ldots, e_d\}$, is called the *associated filtration* of the chain $N_1, \ldots, N_d$.

Let $m_i \in \wedge^i M$ for $1 \le i \le d$. We say the sequence $m_1, \ldots, m_d$ forms a *chain* in M if the sequence $Am_1, \ldots, Am_d$ forms a chain. In this case, we call the associated filtration of $Am_1, \ldots, Am_d$ the *associated filtration* of the chain $m_1, \ldots, m_d$.

The following lemma is a simple exercise in linear algebra.

Lemma 5.35. *The sequence $m_1, \ldots, m_d$ forms a chain in M if and only if the following hold.*

(1) *m_1 is cofree.*

(2) $m_1 \wedge m_i = 0$ for $2 \le i \le d$.

(3) *The sequence of quotients of $m_2, \ldots, m_d$ by m_1 forms a chain in M/Am_1.*

Lemma 5.36. *Suppose A is a Bézout domain, and let M be a free A-module of rank d. For each $1 \le i \le d$, let $m_i \in \wedge^i M$ be cofree. Now suppose $A \to B$ is an injective map of commutative rings. Then the sequence $m_1, \ldots, m_d$ forms a chain in M if and only if it forms a chain in $M \otimes_A B$.*

Proof. To show the "if" part of the lemma, we proceed by induction on d. The initial case is trivial. Suppose it is true for $d = k - 1$ for some $k \ge 2$. Now suppose rank $M = k$ and the sequence $m_1, \ldots, m_k$ forms a chain in $M \otimes_A B$. Then $m_1 \wedge m_i = 0$ in $\wedge^{i+1}(M \otimes_A B)$. Hence $m_1 \wedge m_i = 0$ in $\wedge^{i+1} M$ since the natural map

$$\wedge^{i+1} M \to \wedge^{i+1}(M \otimes_A B)$$

is injective. Furthermore, since m_i is cofree in $\wedge^i M$, its quotient by m_1 is cofree in $\wedge^{i-1}(M/Am_1)$ by the Bézout property of A. We therefore conclude the lemma from Lemma 5.35 and the inductive assumption. $\square$

Lemma 5.37. *Let L be a finite extension of $\mathbb{Q}_p$, and let D be a (φ, Γ)-module over $\mathcal{R}_{L'}$ of rank d (recall that $L' = L \otimes_{\mathbb{Q}_p} K_0'$). Then the following are true.*

(1) *Let D_1 be a rank 1 (φ, Γ)-submodule of D. Then D_1 is cofree in D if and only if D_1^r is cofree in D^r for some (hence all) sufficiently small r.*

(2) *For $1 \le i \le d$, let D_i be a rank 1 (φ, Γ)-submodule of $\wedge^i D$. Then the sequence $D_1, \ldots, D_d$ forms a chain in D if and only if the sequence $D_1^r, \ldots, D_d^r$ forms a chain in D^r for some (hence all) sufficiently small r.*

Proof. We deduce (1) from Lemma 5.25. We deduce (2) from (1) and Lemma 5.36. $\square$

Now let S be an affinoid algebra over $\mathbb{Q}_p$.

Lemma 5.38. *Let I be a closed subinterval of $(0, \infty)$, and let M_S^I be a vector bundle over $\mathcal{R}_{K_0'}^I \widehat{\otimes}_{\mathbb{Q}_p} S$ of rank d. For $1 \le i \le d$, let a_i be a global section of $\wedge^i M_S^I$ such that its image in $\wedge^i M_x^I$ is cofree for any $x \in M(S)$. Then the set of $x \in M(S)$ where the image of the sequence $a_1, \ldots, a_d$ forms a chain in M_x^I forms a reduced Zariski closed subspace of $M(S)$.*

Proof. We proceed by induction on d. The case $d = 1$ is trivial. Now suppose that the lemma is true for $d = k - 1$ for some $k \ge 2$, and that M_S^I has rank k. By assumption, the image of a_1 in M_x^I is cofree for any $x \in M(S)$. Hence $M_S^I / (\mathcal{R}_{K_0'}^I \widehat{\otimes}_{\mathbb{Q}_p} S)a_1$ is a vector bundle of rank $k - 1$ over $\mathcal{R}_{K_0'}^I \widehat{\otimes}_{\mathbb{Q}_p} S$.

Since I is a closed interval, using Lemma 5.17, we may suppose that both M_S^I and $M_S^I / (\mathcal{R}_{K_0'}^I \widehat{\otimes}_{\mathbb{Q}_p} S)a_1$ are free over $\mathcal{R}_{K_0'}^I \widehat{\otimes}_{\mathbb{Q}_p} S$ by restricting on a finite covering

of $M(S)$ by affinoid subdomains. Thus a_1 is cofree in M_S^I. Extend $\{a_1\}$ to a basis of M_S^I over $\mathcal{R}_S^I$. We may expand a_i using this basis. It is then straightforward to see that for each $2 \le i \le k$, the set of $x \in M(S)$, where the image of $a_1 \wedge a_i$ in $\wedge^{i+1} M_x^I$ is zero, forms a reduced Zariski closed subspace of $M(S_i)$ of $M(S)$. Furthermore, it follows that

$$a_1 \wedge a_i = 0$$

in $\wedge^{i+1} M_{S_i}^I$ for $2 \le i \le d$.

Now let $M(S')$ be the intersection of all $M(S_i)$, and let

$$b_i \in \wedge^{i-1}(M_{S'}^I/(\mathcal{R}_{K_0'}^I \widehat{\otimes}_{\mathbb{Q}_p} S')a_1)$$

be the quotient of a_i by a_1. For $x \in M(S')$, since the image of a_i is cofree in $\wedge^i M_x^I$, the image of b_i in $\wedge^{i-1}(M_x^I/(\mathcal{R}_{K_0'}^I \otimes_{\mathbb{Q}_p} k(x))a_1(x))$ is cofree by the Bézout property of $\mathcal{R}_{K_0'}^I$. By Lemma 5.35, the desired subset of $M(S)$ is then the set of x where the image of the sequence $b_2, \ldots, b_k$ forms a chain in $(M_{S'}^I/(\mathcal{R}_{K_0'}^I \widehat{\otimes}_{\mathbb{Q}_p} S')a_1)_x$. We therefore conclude the case $d = k$ by the inductive assumption. $\qquad\square$

Lemma 5.39. *Let D_S be a (φ, Γ)-module over $\mathcal{R}_{K_0'} \widehat{\otimes}_{\mathbb{Q}_p} S$ of rank d. For $1 \le i \le d$, let $D_i \subset \wedge^i D_S$ be a rank 1 (φ, Γ)-submodule over $\mathcal{R}_{K_0'} \widehat{\otimes}_{\mathbb{Q}_p} S$. If D_i specializes to a rank 1 cofree $\mathcal{R}_{K_0'} \otimes_{\mathbb{Q}_p} k(x)$-submodule of $\wedge^i D_x$ for any $x \in M(S)$ and $1 \le i \le d$, the set of $x \in M(S)$ where the image of the sequence $D_1, \ldots, D_d$ forms a chain in D_x forms a reduced Zariski closed subspace of $M(S)$.*

Proof. By Lemma 5.37(2), the sequence $D_1, \ldots, D_d$ forms a chain in D_x if and only if $D_1^r, \ldots, D_d^r$ forms a chain in D_x^r for all sufficiently small r. By Lemma 5.36, the latter holds if and only if $D_1^{[r,r]}, \ldots, D_d^{[r,r]}$ forms a chain in $D_x^{[r,r]}$. We then deduce the lemma by Lemma 5.38. $\qquad\square$

In the following, for $1 \le i \le d$, set

$$N_{i,X} = \mathcal{D}_{\mathrm{rig}}^\dagger((\wedge^i V_X)(\eta_i^{-1}))^{\varphi^f = \alpha_i, \Gamma = 1} \otimes_{K_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_X} \mathcal{D}_{\mathrm{rig}}^\dagger(\eta_i),$$

which is a rank 1 (φ, Γ)-submodule of $\mathcal{D}_{\mathrm{rig}}^\dagger(\wedge^i V_X)$ of type η_i. Recall that in §1.3 for $1 \le i \le d$, we define the character $\delta_i : K^\times \to \mathcal{O}(X)^\times$ by setting $\delta_i|_{\mathcal{O}_K^\times} = \eta_i$ and $\delta_i(\pi_K) = \alpha_i$.

Proposition 5.40. *The triangulation locus of V_X forms a reduced Zariski closed subspace of X_s. Furthermore, the sequence $N_1, \ldots, N_d$ gives rise to a global triangulation of V_X on the triangulation locus. That is, for any affinoid subdomain $M(S)$ of the triangulation locus, the sequence $N_1, \ldots, N_d$ forms a chain in $\mathrm{D}_{\mathrm{rig}}^\dagger(V_S)$ whose associated filtration is a triangulation of $\mathrm{D}_{\mathrm{rig}}^\dagger(V_S)$ with parameters $(\delta_i/\delta_{i-1})_{1 \le i \le d}$.*

Proof. Note that by its definition, the triangulation locus of V_X is exactly the set of $x \in X_s$ where the image of the sequence $N_1, \ldots, N_d$ forms a chain in $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$. We then deduce the first statement from Lemma 5.39.

Now let $M(S)$ be an affinoid subdomain of the triangulation locus of V_X, and suppose $\mathrm{D}^{\dagger,s}_{\mathrm{rig}}(V_S)$ is defined for some $s > 0$. Set $N^{(0)}_{i,X} = N_{i,X}$. Since $N^{(0)}_{1,x}$ is a rank 1 saturated (φ, Γ)-submodule of $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$ for $x \in M(S)$, by Proposition 5.26,

$$D^{(1)}_S = \mathrm{D}^\dagger_{\mathrm{rig}}(V_S)/N^{(0)}_{1,S}$$

is a (φ, Γ)-module of rank $d-1$ over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ with a representative vector bundle

$$D^{(1),\rho(s)}_S = \mathrm{D}^{\dagger,s}_{\mathrm{rig}}(V_S)/N^{(0),\rho(s)}_{1,S}.$$

By Lemma 5.17, we choose a finite covering $\{M(S_j)\}_{j \in J}$ of $M(S)$ by affinoids such that all the vector bundles $N^{(0),[\rho(s)/p^f,\rho(s)]}_{i,S_j}$ and $D^{(1),[\rho(s)/p^f,\rho(s)]}_{S_j}$ are free. Since $M(S)$ is contained in the triangulation locus, it follows that

$$N^{(0),[\rho(s)/p^f,\rho(s)]}_{1,S_j} \wedge N^{(0),[\rho(s)/p^f,\rho(s)]}_{i,S_j} = 0$$

for $2 \le i \le d$ and $j \in J$. Taking the quotient of $N^{(0),[\rho(s)/p^f,\rho(s)]}_{i,S_j}$ by $N^{(0),[\rho(s)/p^f,\rho(s)]}_{1,S_j}$ for each j and gluing these quotients, we obtain a vector bundle $N^{(1),[\rho(s)/p^f,\rho(s)]}_{i,S}$ over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$. Furthermore, note that each $N^{(1),[\rho(s)/p^f,\rho(s)]}_{i,S_j}$ admits a basis e satisfying

$$\varphi^f(e) = (\alpha_i/\alpha_1)(e).$$

Therefore, we can extend $N^{(1),[\rho(s)/p^f,\rho(s)]}_{i,S}$ to a rank 1 (φ, Γ)-submodule, which is of type δ_i/δ_1, of $D^{(1)}_S$.

We may iterate the above procedure as follows. Suppose after the k-th step, we have a (φ, Γ)-module $D^{(k)}_S$ over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ of rank $d-k$ and a rank 1 (φ, Γ)-submodule $N^{(k)}_{i,S}$ of $\wedge^{i-k} D_i$ of type δ_i/δ_k, which specializes to a saturated (φ, Γ)-submodule of $\wedge^{i-k} D^i_x$ for any $x \in X$, for each $k+1 \le i \le d$. Now let

$$D^{(k+1)}_S = D^{(k)}_S/N^{(k)}_{k+1,S}.$$

It is then a (φ, Γ)-module over $\mathcal{R}_{K'_0} \widehat{\otimes}_{\mathbb{Q}_p} S$ of rank $d-k-1$ by Proposition 5.26. Then by the same argument as above, for each $k+2 \le i \le d$, we get a rank 1 (φ, Γ)-submodule $N^{(k+1)}_{i,S}$ of $\wedge^{i-k-1} D^{(k+1)}_S$ of type δ_i/δ_{k+1}, which specializes to saturated (φ, Γ)-submodule of $\wedge^{i-k-1} D^{(k+1)}_x$ for $x \in M(S)$.

Now let $\mathrm{Fil}_i(\mathrm{D}^\dagger_{\mathrm{rig}}(V_S)) = \ker(\mathrm{D}^\dagger_{\mathrm{rig}}(V_S) \to D_S^{(i)})$ for $1 \le i \le d$. It follows from the above procedure that $(\mathrm{Fil}_i(\mathrm{D}^\dagger_{\mathrm{rig}}(V_S)))_{1 \le i \le d}$ is a triangulation of $\mathrm{D}^\dagger_{\mathrm{rig}}(V_S)$ with successive quotients

$$\mathrm{Fil}_{i+1}(\mathrm{D}^\dagger_{\mathrm{rig}}(V_S))/\mathrm{Fil}_i(\mathrm{D}^\dagger_{\mathrm{rig}}(V_S)) \cong N^{(i)}_{i+1,S}$$

for $0 \le i \le d-1$. The yields the second statement of the theorem. $\qquad\square$

It remains to show that the triangulation locus contains all regular non-critical points.

Proposition 5.41. *For $x \in X$, if V_x satisfies*

$$\dim_{k(x)} \mathrm{D}^\dagger_{\mathrm{rig}}((\wedge^i V_x^{\mathrm{ss}})(\eta_i(x)^{-1}))_\sigma^{\varphi^f = \alpha_i(x), \Gamma = 1} = 1$$

for all $1 \le i \le d-1$ and $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, and $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$ admits a triangulation $(\mathrm{Fil}_i(\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)))_{1 \le i \le d}$ with parameters $(\delta_i/\delta_{i-1})(x)_{1 \le i \le d}$, then the sequence

$$(\mathscr{D}^\dagger_{\mathrm{rig}}((\wedge^i V_X)(\eta_i^{-1}))^{\varphi^f = \alpha_i, \Gamma = 1}_\sigma \otimes_{K_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_X} \mathrm{D}^\dagger_{\mathrm{rig}}(\eta_i(x)))_{1 \le i \le d}$$

forms a chain in $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$ and its associated filtration is just $(\mathrm{Fil}_i)_{1 \le i \le d}$.

Proof. By Proposition 5.13, $(\mathscr{D}^\dagger_{\mathrm{rig}}((\wedge^i V_X)(\eta_i^{-1}))_\sigma^{\varphi^f = \alpha_i, \Gamma = 1}$ is locally free of rank 1 around x, and

$$(\mathscr{D}^\dagger_{\mathrm{rig}}((\wedge^i V_X)(\eta_i^{-1}))_\sigma^{\varphi^f = \alpha_i, \Gamma = 1} \otimes k(x) \to \mathrm{D}^\dagger_{\mathrm{rig}}((\wedge^i V_x)(\eta_i(x)^{-1}))_\sigma^{\varphi^f = \alpha_i(x), \Gamma = 1}$$

is an isomorphism for all i and σ. Thus

$$(\mathscr{D}^\dagger_{\mathrm{rig}}((\wedge^i V_X)(\eta_i^{-1}))^{\varphi^f = \alpha_i, \Gamma = 1} \otimes k(x) \to \mathrm{D}^\dagger_{\mathrm{rig}}((\wedge^i V_x)(\eta_i(x)^{-1}))^{\varphi^f = \alpha_i(x), \Gamma = 1}$$

is an isomorphism. By assumption, $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$ admits a triangulation $(\mathrm{Fil}_i)_{1 \le i \le d}$ with parameters $(\delta_i/\delta_{i-1})_{1 \le i \le d}$. In particular, $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$ contains a rank 1 (φ, Γ)-submodule

$$D = \mathrm{Fil}_1(\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)) \cong \mathcal{R}_{k(x)}(\delta_1).$$

Recall that $\mathcal{R}_{k(x)}(\delta_1)$ is defined to be

$$D_{\alpha_1(x)} \otimes_{K_0 \otimes_{\mathbb{Q}_p} k(x)} \mathrm{D}^\dagger_{\mathrm{rig}}(\eta_1(x)).$$

We then deduce

$$\dim_{k(x)}(D(\eta_1^{-1}(x)))_\sigma^{\varphi^f = \alpha_1(x), \Gamma = 1} \ge 1.$$

This forces

$$(D(\eta_1^{-1}(x)))_\sigma^{\varphi^f = \alpha_1(x), \Gamma = 1} = \mathrm{D}^\dagger_{\mathrm{rig}}(V_x(\eta_1^{-1}(x)))_\sigma^{\varphi^f = \alpha_1(x), \Gamma = 1}$$

for all σ.

Hence

$$(D(\eta_1^{-1}(x)))^{\varphi^f=\alpha_1(x),\Gamma=1} = \mathrm{D}_{\mathrm{rig}}^\dagger(V_x(\eta_1^{-1}(x)))^{\varphi^f=\alpha_1(x),\Gamma=1}.$$

It follows that the image of the map

$$\mathscr{D}_{\mathrm{rig}}^\dagger((\wedge^i V_X)(\eta_1^{-1}))^{\varphi^f=\alpha_1,\Gamma=1} \otimes_{K_0\otimes_{\mathbb{Q}_p}k(x)} \mathrm{D}_{\mathrm{rig}}^\dagger(\eta_1(x)) \to \mathrm{D}_{\mathrm{rig}}^\dagger(V_x)$$

is exactly $\mathrm{Fil}_1(\mathrm{D}_{\mathrm{rig}}^\dagger(V_x))$. By a similar argument, we deduce that the image of the map

$$\mathscr{D}_{\mathrm{rig}}^\dagger((\wedge^i V_X)(\eta_i^{-1}))^{\varphi^f=\alpha_i,\Gamma=1} \otimes_{K_0\otimes_{\mathbb{Q}_p}k(x)} \mathrm{D}_{\mathrm{rig}}^\dagger(\eta_i(x)) \to \wedge^i \mathrm{D}_{\mathrm{rig}}^\dagger(V_x)$$

is exactly $\wedge^i\mathrm{Fil}_i(\mathrm{D}_{\mathrm{rig}}^\dagger(V_x))$. This yields the desired result. $\square$

Theorem 5.42. *The triangulation locus of V_X contains all the points which satisfy the assumption of Proposition 5.41. In particular, the triangulation locus of V_X contains all regular non-critical points. As a consequence, the triangulation locus of V_X coincides with the saturated locus X_s, which is a Zariski open and dense subspace of X, and the (φ,Γ)-modules*

$$\mathrm{D}_{\mathrm{rig}}^\dagger((\wedge^i V_S)(\eta_i^{-1}))^{\varphi^f=\alpha_i,\Gamma=1} \otimes_{K_0\otimes_{\mathbb{Q}_p}S} \mathrm{D}_{\mathrm{rig}}^\dagger(\eta_i)$$

for $1 \le i \le d$ give rise to a triangulation of $\mathrm{D}_{\mathrm{rig}}^\dagger(V_S)$ with parameters $(\delta_i/\delta_{i-1})_{1\le i\le d}$ on any affinoid subdomain $M(S)$ of X_s.

Proof. The first assertion is an immediate consequence of Proposition 5.41. Furthermore, it is clear that regular non-critical points satisfy the assumption of Proposition 5.41. Thus they belong to the triangulation locus of V_X. On the other hand, note that X_s is the intersections of the saturated loci of the weakly refined families $\wedge^i V_X$ for all $1 \le i \le d$. Hence it is Zariski open by Propositions 5.6. Since the set of regular non-critical points is Zariski dense in X, it follows that it is Zariski dense in X_s, and X_s is Zariski dense in X. We then conclude the rest of the theorem by Proposition 5.40. $\square$

As mentioned in the introduction, it is expected that all non-critical points belong to the locus of global triangulation. Regarding this point, we make the following conjecture.

Conjecture 5.43. *For $x \in X$, if $\mathrm{D}_{\mathrm{rig}}^\dagger(V_x)$ admits a triangulation $(\mathrm{Fil}_i(\mathrm{D}_{\mathrm{rig}}^\dagger(V_x)))_{1\le i\le d}$ with parameters $((\delta_i/\delta_{i-1})(x))_{1\le i\le d}$ such that*

$$\dim_{k(x)}(\mathrm{D}_{\mathrm{rig}}^\dagger(V_x)/\mathrm{Fil}_{i-1}(\mathrm{D}_{\mathrm{rig}}^\dagger(V_x)))_\sigma^{\varphi^f=\alpha_i/\alpha_{i-1},\Gamma=\eta_i/\eta_{i-1}} = 1$$

for all $1 \le i \le d$ and $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, then x belongs to the triangulation locus.

5.4. Specializations of refined families.

Lemma 5.44. *Let L be a finite extension of E, and let D be a (φ, Γ)-module over $\mathcal{R}_{L'}$ ($L' = L \otimes_{\mathbb{Q}_p} K_0'$). Let D_1 be a (φ, Γ)-submodule of D, and let D_1' be its saturation in D. Then there exists a positive integer k such that $t^k D_1' \subset D_1$.*

Proof. Note that $\mathrm{Ind}_{\Gamma_K}^{\Gamma_{\mathbb{Q}_p}} D_1'$ is the saturation of $\mathrm{Ind}_{\Gamma_K}^{\Gamma_{\mathbb{Q}_p}} D_1$ in $\mathrm{Ind}_{\Gamma_K}^{\Gamma_{\mathbb{Q}_p}} D$. By [26, Proposition 3.1], there exists a positive integer k such that $t^k \mathrm{Ind}_{\Gamma_K}^{\Gamma_{\mathbb{Q}_p}} D_1' \subset \mathrm{Ind}_{\Gamma_K}^{\Gamma_{\mathbb{Q}_p}} D_1$. This yields the lemma. $\qquad\square$

Theorem 5.45. *For $x \in X$, the p-adic representation V_x is trianguline.*

Proof. Let $M(S)$ be an affinoid neighborhood of x. By the same argument as in the proof of Proposition 5.13, there exists a proper birational map $\pi : X' \to M(S)$ such that for all $1 \le i \le d$ and $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, the coherent sheaves

$$\mathscr{D}_{\mathrm{rig}}^\dagger((\wedge^i V_{X'})(\pi^*(\eta_i^{-1})))_\sigma^{\varphi^f = \pi^*(\alpha_i), \Gamma = 1}$$

are locally free of rank 1 and the natural map

$$\mathscr{D}_{\mathrm{rig}}^\dagger((\wedge^i V_{X'})(\pi^*(\eta_i^{-1})))_\sigma^{\varphi^f = \pi^*(\alpha_i), \Gamma = 1} \otimes k(x') \to \mathrm{D}_{\mathrm{rig}}^\dagger((\wedge^i V_{x'})(\pi^*(\eta_i^{-1})(x')))_\sigma$$

is injective for any $x' \in X'$. By the previous lemma,

$$\mathscr{D}_{\mathrm{rig}}^\dagger((\wedge^i V_{X'})(\pi^*(\eta_i^{-1})))^{\varphi^f = \pi^*(\alpha_i), \Gamma = 1} \otimes_{K_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_{X'}} \mathscr{D}_{\mathrm{rig}}^\dagger(\pi^*(\eta_i))[1/t]$$

specializes to a rank 1 saturated $(\mathcal{R}_{K_0'} \otimes_{\mathbb{Q}_p} k(x))[1/t]$-submodule in $\mathrm{D}_{\mathrm{rig}}^\dagger(\wedge^i V_{x'})[1/t]$.

Pick some $s \ge s(V_S)$ such that $s \ne r_n$ for any $n \in \mathbb{N}$, and put $r = \rho(s)$. It follows that t has no roots in the annulus $v_p(T) = r$. Thus t is invertible in $\mathcal{R}_{K_0'}^{[r,r]}$. Hence the natural map

$$\mathcal{R}_{K_0'}^r[1/t] \to \mathcal{R}_{K'}^{[r,r]}$$

is injective. Now let $M(S')$ be an affinoid subdomain of X'. Set $D_{S'} = \mathrm{D}_{\mathrm{rig}}^\dagger(V_{S'})$ and $D_{i,S'} = \mathrm{D}_{\mathrm{rig}}^\dagger(\pi^*(\eta_i)|_{M(S')})$ for $1 \le i \le d$. We claim that the sequence

$$(\mathrm{D}_{\mathrm{rig}}^\dagger((\wedge^i V_{S'})(\pi^*(\eta_i^{-1})))^{\varphi^f = \pi^*(\alpha_i), \Gamma = 1} \otimes_{K_0 \otimes_{\mathbb{Q}_p} S'} D_{i,x}^{[r,r]})_{1 \le i \le d}$$

forms a chain in $D_{x'}^{[r,r]}$ for any $x' \in M(S')$. In fact, by Lemma 5.38, the set of such x' forms a reduced Zariski closed subspace of $M(S')$. On the other hand, suppose π is an isomorphism on a Zariski dense and open subset U of X'. By Theorem 5.42, after shrinking U, we may further suppose that $\pi(U)$ is contained in the triangulation locus of $M(S)$. It is then clear that the sequence

$$(\mathrm{D}_{\mathrm{rig}}^\dagger((\wedge^i V_{S'})(\pi^*(\eta_i^{-1})))^{\varphi^f = \pi^*(\alpha_i), \Gamma = 1} \otimes_{K_0 \otimes_{\mathbb{Q}_p} S'} D_{i,x}^{[r,r]})_{1 \le i \le d}$$

forms a chain in $D_{x'}^{[r,r]}$ for any $x' \in U \cap M(S')$. Since $U \cap M(S')$ is Zariski dense in $M(S')$, the claim follows.

The claim and Lemma 5.37 then imply that the image of the sequence

$$(D^\dagger_{rig}((\wedge^i V_{S'})(\pi^*(\eta_i^{-1})))^{\varphi^f = \pi^*(\alpha_i), \Gamma = 1} \otimes_{K_0 \otimes_{\mathbb{Q}_p} S'} D^r_{i,S'}[1/t])_{1 \leq i \leq d}$$

forms a chain in $D^{\dagger,s}_{rig}(V_{x'})[1/t]$ for any $x' \in M(S')$. Hence $D^\dagger_{rig}(V_{x'})[1/t]$ is triangulable (in the obvious sense). This yields that $D^\dagger_{rig}(V_{x'})$ is triangulable. It remains to show that $D^\dagger_{rig}(V_x)$ is triangulable. Without loss of generality we may assume that $k(x')$ is Galois over $k(x)$ for some x' in the preimage of x. In this case, define a $G = \mathrm{Gal}(k(x')/k(x))$-action on

$$D^\dagger_{rig}(V_{x'}) = D^\dagger_{rig}(V_x) \otimes_{k(x)} k(x')$$

by setting $g(a \otimes b) = a \otimes g(b)$. It then follows that the triangulation of $D^\dagger_{rig}(V_{x'})$ maps onto a triangulation of $D^\dagger_{rig}(V_x)$ via the projection

$$\frac{1}{|G|} \sum_{g \in G} g : D^\dagger_{rig}(V_{x'}) \to D^\dagger_{rig}(V_x). \qquad \square$$

5.5. Application to the eigencurve. Fix a positive integer N which is prime to p. Let S be the set of places of $\mathbb{Q}$ consisting of the infinite place and the places dividing pN. Let $\overline{V}$ be a two dimensional $G_{\mathbb{Q},S}$-representation over a finite field of characteristic p, which is p-modular in the sense of [11]. Let $R_{\overline{V}}$ be the universal deformation ring of the pseudo representation associated to $\overline{V}$. Let $X_{\overline{V}}$ be the generic fiber of $\mathrm{Spf}(R_{\overline{V}})$, which is a rigid analytic space over $\mathbb{Q}_p$. By the works of Coleman–Mazur [11] and Buzzard [9], there is a $\mathbb{Q}_p$-rigid analytic curve $\mathcal{C} \subset X_{\overline{V}} \times \mathbb{G}_m$ whose $\mathbb{C}_p$-valued points correspond bijectively to overconvergent eigenforms of tame level N, which are of finite slope, and whose residual Galois representation have the same semi-simplification as $\overline{V}$. We further assume that $\mathcal{C}$ belongs to the cuspidal part of the eigencurve. That is, the overconvergent modular forms parametrized by $\mathcal{C}$ are all cuspidal.

Let

$$T : G_{\mathbb{Q},S} \to \mathcal{O}(\mathcal{C})$$

be the pseudo representation obtained by pulling back the universal pseudo representation of $G_{\mathbb{Q},S}$ on $X_{\overline{V}}$ via the composite

$$\mathcal{C} \to X_{\overline{V}} \times \mathbb{G}_m \to X_{\overline{V}}. \qquad (5.7)$$

Let $\alpha \in \mathcal{O}(\mathcal{C})^\times$ denote the function of U_p-eigenvalue. Let $\kappa : \mathcal{C} \to \mathcal{W}$ be the weight map. We normalize κ in such a way that if $x \in \mathcal{C}$ is a classical eigenform of weight k, then $\kappa(x) = k - 1$.

Let $\widetilde{\mathcal{C}}$ denote the normalization of $\mathcal{C}$. By [11], there exists a family of p-adic representations of $G_{\mathbb{Q},S}$ of dimension 2 over $\mathcal{C}$ whose associated pseudo

representation is isomorphic to the pullback of T via $\widetilde{\mathcal{C}} \to \mathcal{C}$. Let $V_{\widetilde{\mathcal{C}}}$ be the dual of this family of p-adic representations. Let $\widetilde{\alpha} \in \mathcal{O}(\widetilde{\mathcal{C}})^{\times}$ denote the pullback of α via $\widetilde{\mathcal{C}} \to \mathcal{C}$. Let $\widetilde{\kappa} : \widetilde{\mathcal{C}} \to \mathcal{C}$ be the composite of κ with $\widetilde{\mathcal{C}} \to \mathcal{C}$. Let $\widetilde{Z}$ be the set of classical points $z \in \mathcal{C}$ such that V_z is crystalline with distinct crystalline Frobenius eigenvalues. By Coleman's classicality theorem, it is straightforward to see that $V_{\widetilde{\mathcal{C}}}$ is a family of 2-dimensional weakly refined p-adic representations together with $\kappa_1 = 0, \kappa_2 = -\widetilde{\kappa}, F = \widetilde{\alpha}, Z = \widetilde{Z}$.

Proposition 5.46. *The coherent sheaf* $\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_{\widetilde{\mathcal{C}}})^{\varphi=F,\Gamma=1}$ *is invertible, and its image in* $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_x)$ *is nonzero for any* $x \in \widetilde{\mathcal{C}}$. *As a consequence,* V_x *is trianguline for any* $x \in \widetilde{\mathcal{C}}$.

Proof. Let $M(S)$ be an affinoid subdomain of $\widetilde{\mathcal{C}}$. Let k be a positive integer such that

$$k > \log_p |F^{-1}|$$

in S. By Theorems 5.3 and 4.12, the map

$$\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S)^{\varphi=F,\Gamma=1} \to (\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))^{\Gamma}$$

is an isomorphism. Note that $(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))^{\Gamma}$ is a finite torsion-free S-module by Proposition 2.23. Hence it is a locally free S-module because S is smooth and 1-dimensional. Thus $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S)^{\varphi=F,\Gamma=1}$ is a locally free S-module. Furthermore, by Proposition 5.6, it is locally free of rank 1 on a Zariski open and dense subspace of $M(S)$. Hence it is locally free of rank 1 on $M(S)$, yielding the first statement of the theorem.

For the second statement, by Proposition 2.23, $(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))/(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))^{\Gamma}$ is finite and torsion-free over S as well. This implies that for any $x \in M(S)$, the natural map

$$(\mathrm{D}_{\mathrm{dif}}^{+,n}(V_S)/(t^k))^{\Gamma} \otimes k(x) \to \mathrm{D}_{\mathrm{dif}}^{+,n}(V_x)/(t^k)$$

is injective. It follows that $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_S)^{\varphi=F,\Gamma=1} \otimes k(x) \to \mathrm{D}_{\mathrm{rig}}^{\dagger}(V_x)$ is injective as well. $\qquad\square$

Proposition 5.47. *For* $x \in \widetilde{\mathcal{C}}$, x *is not saturated if and only if* V_x *satisfies one of the following two disjoint conditions:*

(1) *The weight* $\kappa(x)$ *is a positive integer and* $v_p(F(x)) > \kappa(x)$. *As a consequence,* V_x *belongs to* $\mathscr{S}_{*}^{\mathrm{ng}} \cap \mathscr{S}_{*}^{\mathrm{HT}}$ *in the sense of* [14]; *hence* V_x *is irreducible, Hodge–Tate and non-de Rham. Furthermore, the image of* $t^{-\kappa(x)}\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_{\widetilde{\mathcal{C}}})^{\varphi=F,\Gamma=1}$ *generates a rank 1 saturated* (φ, Γ)-*submodule in* $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_x)$.

(2) *The weight $\kappa(x)$ is a positive integer and $v_p(F(x)) = \kappa(x)$, and V_x has a rank 1 subrepresentation V'_x which is crystalline with Hodge–Tate weight $-\kappa(x)$. Furthermore, in this case, the image of $\mathscr{D}^\dagger_{\mathrm{rig}}(V_{\widetilde{C}})^{\varphi=F,\Gamma=1}$ in $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$ is $k(x) \cdot t^{\kappa(x)} e'$ where e' is a canonical basis of $\mathrm{D}^\dagger_{\mathrm{rig}}(V'_x)$.*

In case (2), if $x \in Z$, then it is critical. Hence V_x is split. Suppose $V_x = V_1 \oplus V_2$ where V_1 has Hodge–Tate weight 0 and V_2 has Hodge–Tate weight $-\kappa(x)$. Then the image of $\mathscr{D}^\dagger_{\mathrm{rig}}(V_{\widetilde{C}})^{\varphi=F,\Gamma=1}$ in $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$ is $k(x) \cdot t^{\kappa(x)} e_2$ where e_2 is a canonical basis of $\mathrm{D}^\dagger_{\mathrm{rig}}(V_2)$.

Proof. Suppose that x is not saturated. Let D be the saturation of the rank 1 (φ, Γ)-submodule of $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$ generated by $\mathscr{D}^\dagger_{\mathrm{rig}}(V_{\widetilde{C}}))^{\varphi=F,\Gamma=1} \otimes k(x)$. Suppose

$$\mathscr{D}^\dagger_{\mathrm{rig}}(V_{\widetilde{C}})^{\varphi=F,\Gamma=1} \otimes k(x) = k(x) \cdot t^k e$$

for some positive integer k and canonical basis e of D. Thus the Hodge–Tate weight of D is $-k$, yielding that $\kappa(x) = k$ is a positive integer. By Kedlaya's slope theory, D has nonnegative slope, yielding that $v_p(\alpha(x)) \geq \kappa(x)$. If the inequality is strict, then V_x satisfies the condition (1). If $v_p(\alpha(x)) = \kappa(x)$, it is straightforward to see that V_x satisfies the condition (2). Furthermore, if $x \in Z$, it is clear that x is critical.

For the converse, suppose V_x satisfies (1). If it is saturated, then it follows from Colmez's classification of 2-dimensional irreducible trianguline representations of $G_{\mathbb{Q}_p}$ [13, §3.3] that $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x)$ belongs to $\mathscr{S}^{\mathrm{ncl}}_+$. However, by [13, Proposition 3.5], we know that all 2-dimensional triangulable (φ, Γ)-modules belonging to $\mathscr{S}^{\mathrm{ncl}}_+$ are non-étale. This makes a contradiction. Now suppose V_x satisfies (2). Note that V_x/V'_x has Hodge–Tate weight 0. Thus if the image of $\mathscr{D}^\dagger_{\mathrm{rig}}(V_{\widetilde{C}})^{\varphi=F,\Gamma=1} \otimes k(x)$ in $\mathrm{D}^\dagger_{\mathrm{rig}}(V_x/V'_x)$ is nonzero, it generates a rank 1 (φ, Γ)-submodule which is of Hodge–Tate weight 0 and positive slope, yielding a contradiction. Therefore $\mathscr{D}^\dagger_{\mathrm{rig}}(V_{\widetilde{C}})^{\varphi=\alpha,\Gamma=1} \otimes k(x)$ maps into $\mathrm{D}^\dagger_{\mathrm{rig}}(V'_x)$. It then follows that the image is of the given form. $\square$

In the case when $\overline{V}$ is an absolutely irreducible $G_{\mathbb{Q},S}$-representation, $R_{\overline{V}}$ coincides with the universal deformation ring of $\overline{V}$[10]. Let $V_{\mathcal{C}}$ be the dual of the pullback of the universal representation of $G_{\mathbb{Q},S}$ on $R_{\overline{V}}$ via (5.7). Let Z be the set of classical points $z \in \mathcal{C}$ such that V_z is crystalline with distinct crystalline Frobenius eigenvalues. Then $V_{\mathcal{C}}$ is a family of 2-dimensional weakly refined p-adic representations over $\mathcal{C}$ together with $\kappa_1 = 0, \kappa_2 = -\kappa, F = \alpha$ and Z. Similarly, we have the following result.

Theorem 5.48. *For any $x \in \mathcal{C}$, $\mathscr{D}^\dagger_{\mathrm{rig}}(V_{\mathcal{C}})^{\varphi=F,\Gamma=1}$ is locally free of rank 1 around x unless $\kappa(x) = 0$, and V^{ss}_x is crystalline and satisfies $\dim D_{\mathrm{crys}}(V^{\mathrm{ss}}_x)^{\varphi=F(x)} = 2$. If x*

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[10]In this case, $\mathcal{C}$ automatically belongs to the cuspidal part of the eigencurve.

*is not of this form, it is not saturated if and only if it satisfies one of the following two
disjoint conditions:*

(1) *The weight $\kappa(x)$ is a positive integer and $v_p(F(x)) > \kappa(x)$. As a
consequence, V_x belongs to $\mathscr{S}_*^{\mathrm{ng}} \cap \mathscr{S}_*^{\mathrm{HT}}$ in the sense of [14]; hence V_x
is irreducible, Hodge–Tate and non-de Rham. Furthermore, in this case
$t^{-\kappa(x)} \mathscr{D}_{\mathrm{rig}}^{\dagger}(V_C)^{\varphi=F,\Gamma=1}$ generates a rank 1 saturated (φ, Γ)-submodule in
$\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_x)$.*

(2) *The weight $\kappa(x)$ is a positive integer and $v_p(F(x)) = \kappa(x)$, and V_x has
a rank 1 subrepresentation V'_x which is crystalline with Hodge–Tate weight
$-\kappa(x)$. Furthermore, in this case, the image of $\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_C)^{\varphi=F,\Gamma=1}$ in $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_x)$
is $k(x) \cdot t^{\kappa(x)} e'$ where e' is a canonical basis of $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V'_x)$.*

(2′) *In case (2), if $x \in Z$, then it is critical. Furthermore, suppose that $V_x = V_1 \oplus
V_2$ where V_1 has Hodge–Tate weight 0 and V_2 has Hodge–Tate weight $-\kappa(x)$.
Then the image of $\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_C)^{\varphi=F,\Gamma=1}$ in $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_x)$ is $k(x) \cdot t^{\kappa(x)} e_2$ where e_2 is
a canonical basis of $\mathrm{D}_{\mathrm{rig}}^{\dagger}(V_2)$.*

Proof. If $\dim_{k(x)} D_{\mathrm{crys}}(V_x^{\mathrm{ss}})^{\varphi=F(x)} \leq 1$, $\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_C)^{\varphi=F,\Gamma=1}$ is locally free of rank 1
around x by Proposition 5.13. Thus if $(\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_C))^{\varphi=F,\Gamma=1}$ is not locally free of
rank 1 around x, then $D_{\mathrm{crys}}(V_x^{\mathrm{ss}})^{\varphi=F(x)}$ is of dimension 2. Furthermore, in this case,
$\kappa(x) = 0$ by the weak admissibility of $D_{\mathrm{crys}}(V_x^{\mathrm{ss}})$. We deduce the rest of the theorem
by the same argument as in the proof of Proposition 5.47.   □

Remark 5.49. By Theorem 5.48, if $\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_C)^{\varphi=F,\Gamma=1}$ is not locally free around x,
then the weak admissibility of $D_{\mathrm{crys}}(V_x^{\mathrm{ss}})$ implies $v_p(F(x)) = 0$; hence x is
ordinary. Furthermore, it follows that the weight character of x is crystalline (hence
unramified) of Hodge–Tate weight 0. By the spectral theory of U_p, we know that the
set of those x is finite.

We conjecture that there is no such x, i.e. $\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_C)^{\varphi=F,\Gamma=1}$ is everywhere
locally free of rank 1 over the eigencurve. In fact, for $p \geq 5$ and $N = 1$, by
virtue of a classical result of Mazur–Wiles [29, §8, Proposition 2], we see that if
the weight character of x is non-trivial on the torsion subgroup of $\mathbb{Z}_p$, V_C is an
extension of a ramified infinite order character by an unramified character around x.
It is then straightforward to see that around x, $\mathscr{D}_{\mathrm{rig}}^{\dagger}(V_C)^{\varphi=F,\Gamma=1}$ is locally free of
rank 1 around x and gives rise to the desired global triangulation. We expect that
an analogue of the result of Mazur–Wiles holds for general p and N; this would
confirm most of our conjecture.

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