Conchoid and Negative Circle

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Seine Antwort lautet: durch den Schönheitssinn. Bevorzugt werden die Kombinationen, die uns gefallen.

Ich will die grosse Bedeutung des Schönheitsgefühls für unser Denken keineswegs leugnen. Was uns gefällt, interessiert uns am meisten, und was uns interessiert, wird vorzugsweise ins Bewusstsein geholt. Aber allein ausschlaggebend ist der ästhetische Sinn nicht. Sonst würden wir ja das Gefühl der Richtigkeit, der unmittelbaren Gewissheit nicht haben, das unsere Einfälle so oft begleitet. Es ist wie beim Wiedererkennen: nicht das schönste Gesicht, nicht der schönste Name sollen herausgeholt werden, sondern der richtige!

Gewiss, schöne und interessante Gesichter merkt man sich leichter, sie drängen sich dem Bewusstsein mehr auf, aber sie werden deswegen noch nicht dem Bewusstsein präsentiert mit dem Gefühl: ja, der ist es.

So ist es auch in der Mathematik. Schöne, symmetrische, elegante Vorstellungskombinationen haben mehr Chance, die richtigen zu sein; sie werden auch leichter aufgefunden, weil sie uns besser gefallen, aber das Gefühl der Sicherheit, das einen Einfall so oft begleitet, hat eine andere Quelle.

Es ist eine Erfahrungstatsache, dass es ein Gefühl der unmittelbaren Evidenz, der intuitiven Sicherheit gibt, in der Mathematik wie im Leben. Dieses Gefühl nun stammt aus dem Unbewussten. Wir sehen einen guten Bekannten und erkennen ihn *sofort* wieder. Das beruht nicht auf einer bewussten Vergleichung mit Erinnerungsbildern, denn diese werden uns gar nicht bewusst. In unser Bewusstsein tritt nur die Überzeugung: Der ist es. Diese Überzeugung muss aber letzten Endes auf einer Vergleichung mit Erinnerungsbildern beruhen. Also findet die Vergleichung im Unbewussten statt.

Das Unbewusste ist also imstande, nicht nur zu assoziieren und zu kombinieren, sondern auch zu *urteilen*. Das Unbewusste urteilt intuitiv, aber das Urteil ist unter günstigen Umständen ganz sicher. Wir wissen: Das ist er, ich habe ihn wiedererkannt.

Kehren wir nun zur mathematischen Erfindung zurück, so können wir zusammenfassend sagen: Das Unbewusste hat drei Richtlinien bei der Auswahl der brauchbaren Vorstellungskombinationen. Erstens wählt es vorzugsweise schöne Kombinationen. Zweitens tritt bei gewissen Begriffsverbindungen die Intuition der Richtigkeit, der Evidenz hinzu. Schliesslich hat das Unbewusste ja vom Bewusstsein einen Auftrag erhalten: die gewünschte Begriffskombination soll Bedingungen erfüllen, die das Bewusstsein gestellt hat. Durch bewusstes Denken habe ich mir klar gemacht: Wenn ich so etwas finden würde, dann wäre die Aufgabe gelöst. Das Unbewusste hat, wie ein gewissenhafter Archivar, etwas gefunden, das die Bedingungen erfüllt, also kann es das Gefundene getrost vorlegen und sagen: Das ist die Lösung.

Conchoid and Negative Circle

1.

In working out problems and constructions in the negative-Euclidean (polar-Euclidean) space determined by an 'absolute point' O functioning as point-at-infinity within, it is convenient to be able to relate this space in a standard way to the

Euclidean space in which it is immersed. (It is assumed that the point O is not in the Euclidean plane-at-infinity ω, and that it carries a spherical 'absolute cone', in perspective with the Euclidean absolute—the *Kugelkreis*.) Where the relation of the two spaces is concerned, we shall describe the Euclidean space as 'positive-Euclidean'. The use of the terms 'positive' and 'negative' in this connection is justified, both by the qualitative relations which arise and by the reciprocal functions ($x = -X^{-1}$, to take the simplest form), in which they find expression. Where no confusion is likely to arise, the respective spaces, also the distances, coordinates, etc. which they determine, will more briefly be designated as 'positive' and 'negative', the qualification 'Euclidean' being understood. Needless to say, in either space, distances and other functions may have positive or negative real values—or again, imaginary values—according to circumstances.

Seen in the light of Projective Geometry and the Cayley-Klein theory, Euclidean space is characterized by the fact that the plane-at-infinity ω is given once for all. If on the other hand a system of coordinates, Cartesian or polar, is required, the origin is free to choose. In negative-Euclidean space the situation is in a way reversed. The point-at-infinity O is not given to us by geometrical intuition. Where the idea is applicable to a phenomenon of Nature[1]), we have to recognize the presence and location of such a point by morphological and dynamic insight. In purely geometrical thinking, we choose it freely. If on the other hand a system of negative, planar or peripheral coordinates is required, the Euclidean plane-at-infinity ω provides a natural planar origin. Often therefore it is convenient to use simultaneous coordinates—pointwise and positive-Euclidean, planar and negative-Euclidean respectively—making the 'negative infinitude' O the origin for the former and the 'positive infinitude' ω the origin for the latter. If then a common unit sphere—a sphere with respect to which O and ω are mutually polar—is chosen, the positive- and negative-Euclidean or pointwise and planar (centric and peripheral) units of distance will be so related that the 'negative distance' between any two planes is automatically equated to the 'positive distance' between their poles with respect to the unit sphere.

The simplest and most harmonious relations are obtained if the unit sphere is imaginary[2]). Due to the fact that an elliptic involution does not alter sense, this has the great advantage that no changes of sign or of direction are required in passing from the one space to the other. In the Cartesian system for example, all the six edges of the fundamental tetrahedron, say $\begin{cases} OXYZ \\ \omega\,\xi\,\eta\,\zeta \end{cases}$, are now functioning as 'axes of coordinates'. The three which meet in O, say x_O, y_O, z_O, are 'pointwise axes' in the ordinary, positive-Euclidean interpretation. The opposite and corresponding three, say $x_\omega, y_\omega, z_\omega$, forming a right-angled triangle (self-polar with respect to the absolute

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[1]) Cf. L. Locher-Ernst, *Projektive Geometrie und die Grundlagen der Euklidischen und Polareuklidischen Geometrie* (Orell Füssli, Zürich 1940), Vorwort, pp. XI–XIV; G. Adams and O. Whicher, *The Living Plant and the Science of Physical and Ethereal Spaces* (Goethean Science Foundation, Clent 1949) and *The Plant between Sun and Earth* (Goethean Science Foundation, Clent 1952).

[2]) In this respect the present treatment differs from Prof. Locher's (l. c., pp. 215–239), where simultaneous coordinates are introduced in such a way as to lead up (in the analogous two-dimensional case, p. 238) to a real unit circle. Our method corresponds to that adopted by Veblen and Young (*Projective Geometry*, Vol. I [Boston (Mass.) 1910], § 61), who relate point and line coordinates so as to lead up to *elliptic* involutions, equivalent to the polar system of an imaginary unit circle. Compare figure 124 and p. 225 in Prof. Locher's text-book with figure 80, pp. 171–2 in Veblen and Young.

circle) in ω, function as 'planar axes' in the chosen origin ω of the negative-Euclidean space. If then the positive direction, for example along the vertical axis z_0, has been chosen to be radially *upward*, with an imaginary unit sphere the positive direction as from the corresponding planar axis z_ω (which will now be the celestial horizon) will also be upward; it will be represented by the movement, inward and *upward from below* towards O, of a sequence of parallel horizontal planes taking their start from ω. So in like manner for the other axes and for all pairs of polar-reciprocal movements. Moreover every analytical equation, representing in point-coordinates a certain spatial form, will by the mere replacement of the symbols represent in plane-coordinates the polar-reciprocal form with respect to the unit sphere. The planar equation has its *direct* interpretation in terms of the negative space with origin ω and point-at-infinity O. If as an outcome of the pointwise equation the Euclidean distance between any two points of the form is, say, δ, the planar equation will express the fact that the negative-Euclidean distance between the corresponding planes of the reciprocal form is also equal, both in sign and magnitude, to δ.

2.

The negative-Euclidean equivalents of translation and of orthogonal reflection (called by Prof. LOCHER in his text-book[1]) *Scherung* and *Fernspiegelung* respectively) are easy to construct. A negative 'translation' will of course be an 'elation', with O and a plane of O as the invariant point and plane. In particular, the finite distance between any two planes, not of O, treated as 'negative-Euclidean vector', can always be 'translated' into an equivalent vector with ω as one or other of its end-planes; the vector is then equal in absolute value to the reciprocal of the Euclidean perpendicular distance of the other end-plane from O. With simultaneous coordinates, many well-known formulae have an immediate and very simple interpretation with regard to the 'negative' component of the dual space. For example the Cartesian

$$\frac{1}{P^2} = \frac{1}{X^2} + \frac{1}{Y^2} + \frac{1}{Z^2}, \tag{1}$$

expressing the perpendicular distance P of a given plane from the origin, is an immediate consequence of the Pythagorean formula

$$r^2 = x^2 + y^2 + z^2, \tag{2}$$

relating the *inward* or negative-Euclidean distance r of the plane, as from the planar origin ω, to its rectangular components.

3.

The present article will apply the theory to one of the most fundamental and necessary constructions—that of the negative-Euclidean sphere and circle and of the consequent 'rotating' movements, where a quick method of relating the problem to its positive-Euclidean setting greatly facilitates the task. For simplicity of illustration we shall begin with the two-dimensional case, namely the theory of the 'negative

[1]) L. LOCHER-ERNST, l. c., pp. 249, 270, 283.

circle'—*Winkelkreis*[1]), in Prof. LOCHER'S terminology—in a plane π containing O, which we may choose to be the (O, X, Y)-plane of the coordinate system. We shall call o the Euclidean line-at-infinity, $\pi\,\omega$, of the chosen plane. Taking O and o as positive and negative origins respectively, the common unit circle, say v^2, may now be written:

$$X^2+Y^2+Z^2=0, \qquad x^2+y^2+z^2=0, \tag{3}$$

in homogeneous point-coordinates $X:Y:Z$ and line-coordinates $x:y:z$, O and o being the point and line $0:0:1$ $(z=0, Z=0)$ respectively. In figure 2 the imaginary circle v^2

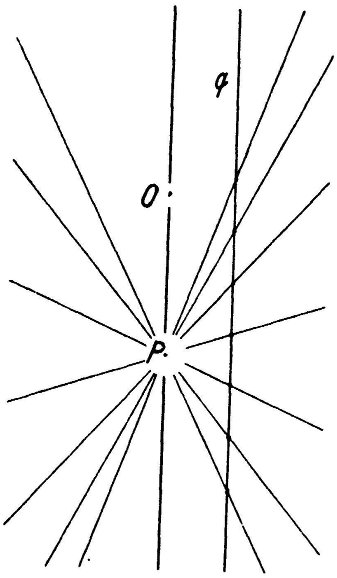

Fig. 1

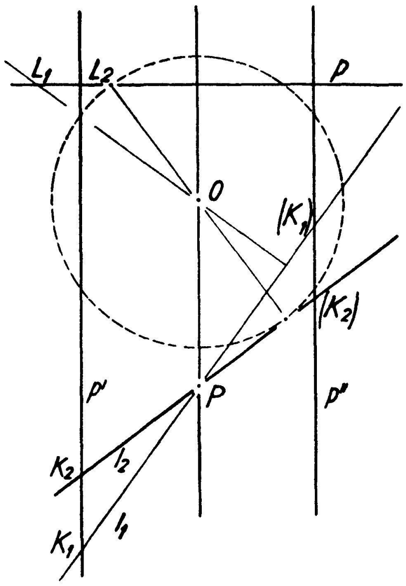

Fig. 2

is indicated by the corresponding ***real*** unit circle. It is the circle with O and o as pole and polar which is transformed into itself by the polarity with respect to v^2.

The negative-Euclidean distance $l_1 l_2$ between any two lines l_1, l_2 of the plane is now equal, both in sign and magnitude, to the positive-Euclidean distance $L_1 L_2$ between their poles with respect to v^2.

In the negative-Euclidean plane (figure 1), a set of equidistant lines of any point P will arise by the harmonic construction of a parabolic or 'step-measure' sequence with respect to the line-at-infinity PO. If q is any line parallel to PO, this parabolic sequence obviously leads in perspective to a positive-Euclidean sequence of equidistant points along q. By suitable choice of the lateral distance of q from PO, the point-to-point and line-to-line distances can be made not only proportional but equal. To this end (figure 2) it is only necessary to rotate the polar line p of P with respect to v^2 through 90° about O into p' or p''. By congruent triangles it is obvious that $K_1 K_2 = L_1 L_2 = l_1 l_2$.

4.

According as the rotation is clockwise or counter-clockwise, two such lines are derivable from any point P of the plane (other than O and the points of o). The transformation $P \rightarrow p'$, or $P \rightarrow p''$, being the product of the polarity v^2 by a 90° rotation,

[1]) L. LOCHER-ERNST, l. c., p. 262.

is clearly a cyclic correlation of period 4. Calling S the 'clockwise' correlation ($P \to p''$), the 'counter-clockwise' will be S^{-1}. We shall define the two resulting lines $p'' = S(P)$ and $p' = S^{-1}(P)$ as the right- and left-hand *satellite lines* of the point P, respectively. Analytically, with the coordinate system oriented in the conventional way, the correlation S, leading to the right-hand satellite p'', is as follows:

$$S: \quad \left.\begin{aligned} \sigma x'' &= +Y, & \tau X'' &= +y, \\ \sigma y'' &= -X, & \tau Y'' &= -x, \\ \sigma z'' &= +Z, & \tau Z'' &= +z. \end{aligned}\right\} \tag{4}$$

$S^2 = S^{-2}$ is the harmonic homology (O, o).

Given the infinities and origins O, o, the relation of a point to its right- or left-hand satellite is a one-valued function of the chosen unit-circle v^2. If the Euclidean unit of distance and with it the radius of v^2 is increased in the proportion α, the lateral distance of the satellite lines from PO, for any given point P, will have to be increased in the proportion α^2. This follows both from the construction by means of the polar line p with respect to v^2 and from the very idea of the measurement of negative distances by positive. In effect, the negative-Euclidean unit of distance will now be *de*creased in the proportion α; hence any given negative distance $l_1 l_2$ (figure 2) will count for α times more, while the pointwise distance $K_1 K_2$ counts for α times less. To compensate for this, the line p' or p'' (q, in figure 1) must therefore be moved α^2 times farther away from PO.

5.

The negative distance between any two lines of the plane can now be quickly and directly measured by the Euclidean distance of the two points, in perspective with them along one or other satellite line of their common point. We apply this to the construction of the *negative-Euclidean circle*. The Euclidean appearance of such a circle depends on the position of the *median line* (*Mittelstrahl des Winkelkreises* in Prof. LOCHER'S terminology), i.e. the polar line of O. If the median line is a line of O, the circle degenerates; if it is o, the 'negative circle' is also a positive-Euclidean circle and no further problem arises. We therefore assume it to be a line c, other than o and not containing O. For convenience, in the following figures we have taken c to be a horizontal line beneath O. With the customary orientation of coordinates it is therefore a line $0:m:1$, or $mY + Z = 0$, where m is a positive real number.

The 'circle' now appears as a conic with O and c as focus and directrix. In form and size it depends on two parameters: (1) the 'negative radius', say r; and (2) the negative or inward distance m of the median line c from the Euclidean line-at-infinity o.

To form the circle directly from the idea of the constant negative distance r of all its tangent lines from c, we have to construct these equal negative distances, inward in either direction as from all the points of c, even as in a positive circle, centre C, we use the constant radius to find the equidistant point-pair outward in either direction along all the lines of C. And as the latter are the 'diameters' of the Euclidean circle, we shall describe the points of c as the *diametral points* of the negative circle. For the construction we now use the satellite lines of all these points in turn. It follows from the correlation (4)—and is readily confirmed by elementary Euclid—that the right-hand satellites of all the points of c emanate from a single point, say N'' (figure 3). It is the point, of which c is the left-hand satellite. Likewise the left-hand satellites emanate from a single point N', of which c is the right-hand satellite. Obviously the line $c_0 = N'N''$ is the parallel to c through O, and the Euclidean distance $N'O = ON'' = m$.

The right-hand satellite of any point D of c is now the line d'' of N'', parallel to DO. Along this line, in either direction as from the point $K'' = cd''$, we mark the points

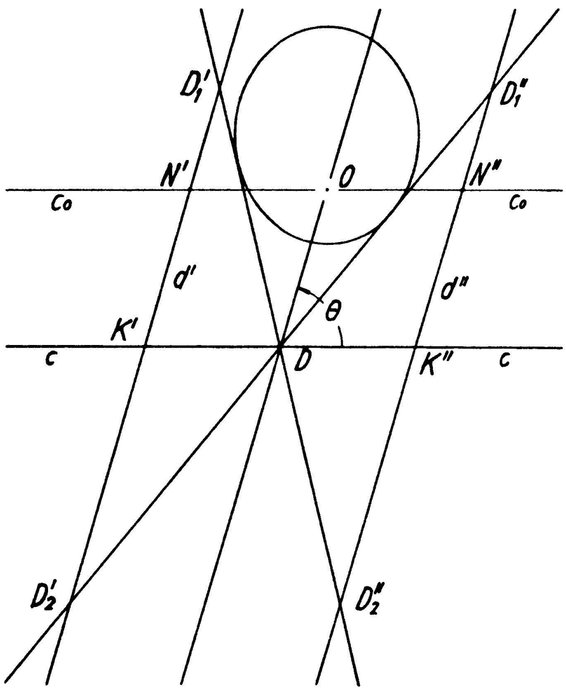

Fig. 3

D_1'', D_2'', making the *Euclidean* distance $K''D_1'' = K''D_2'' = r$. The lines DD_1'', DD_2'' are then the tangents from D to the negative circle, radius r, median line c. Clearly the locus of the end-points D_1'', D_2'' is the *Conchoid of Nicomedes* with node N'', base-line c and 'radial parameter' r (*Strecke* in LORIA's notation[1])). We shall describe it as an '***auxiliary conchoid***', with the help of which the given negative circle can be formed.

The same construction can be made, using the left-hand satellites of the points of c, emanating from the point N' on the other side of O. Therefore the tangents of the required conic must be the lines joining corresponding pairs of points, respectively above and below c, on either of two congruent conchoids to left and right of O; these are transformed into one-another by the lateral translation $N'N'' = 2\,m$. Expressing the same fact in another way, the conic is enveloped by the diagonals of a mobile parallelogram $D_1'D_2'D_2''D_1''$, of which the sides are of fixed length $2\,r$, $2\,m$, respectively. The movement has one degree of freedom—the angle, say θ, between the pairs of sides—and is subject to these conditions: (1) the bisecting line to which the sides

[1]) G. LORIA, *Spezielle algebraische und transcendente ebene Kurven* (Teubner, Leipzig 1902), § 66, p. 128.

$2m$ are parallel is the fixed line c; and (2) the other pair of sides, of length $2r$, rotate about the fixed points N', N''.

According as $m < r$ (figure 3), $m = r$ (figure 4), or $m > r$, the negative circle appears, in its positive-Euclidean aspect, as an ellipse, a parabola or a hyperbola. For the parabola (figure 4), the parallelogram becomes a rhomb, and in its central, symmetrical position a square. To avoid undue complication, the horizontal sides of the parallelograms have been omitted in the figures.

6.

The whole can obviously be reproduced by a mechanical contrivance—a kind of *negative compasses*, causing the lines $D'_1D''_2$, $D''_1D'_2$, whose common point D moves

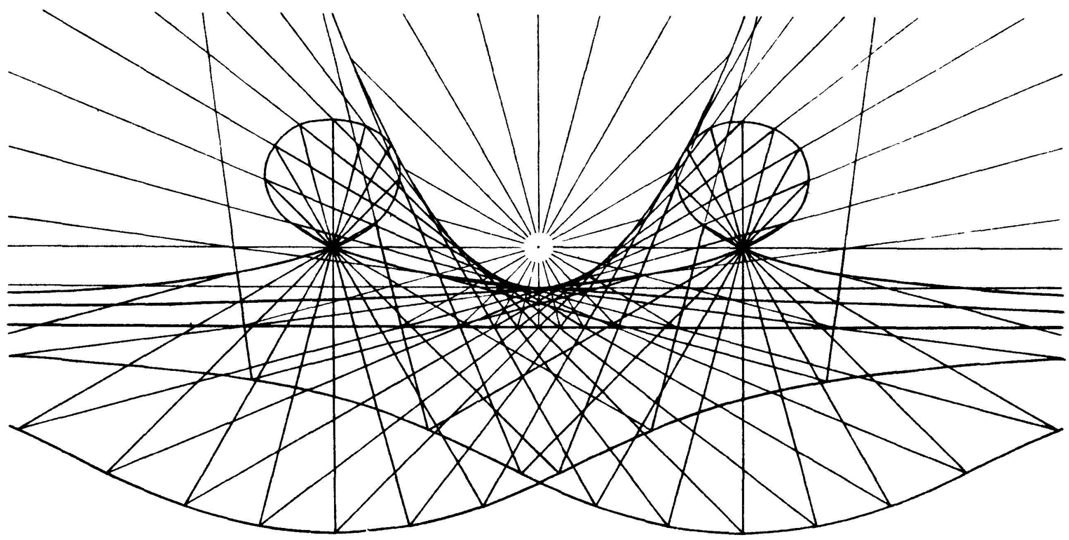

Fig. 4

as 'diametral point' along the median line c, to turn in such a way as to ***mould*** the required 'circle'. In practice it will only not be possible to obtain by this method the portions of the curve near the ends of the major axis, for as these regions are approached the diametral point D moves out into the Euclidean infinite. In principle, however, the construction is complete. Such a contrivance, with the sliding line c and pivots N', N'' fixed and the lengths $D'_1D'_2 = D''_1D''_2 = 2r$ adjustable, will serve to mould a whole family of 'concentric' (or rather, 'commedial') negative circles with common median line c and varying radius r. These are of course, in their Euclidean aspect, a family of conics with common focus and directrix O, c. The ellipses tending inward towards O represent the largest, and the hyperbolae flattening into c the smallest of the 'negative circles'.

Whilst the diagonals of the parallelogram are forming the conic linewise, the four corner-points, D'_1, etc., are describing a pair of equal and parallel conchoids by their pointwise movement (figure 4). The classical construction of the *Conchoid of Nicomedes* acquires a new significance. For it has obvious analogies to the construction of the circle as curve of constant radius. In both instances a pair of points, at a *constant outward distance* in either of two opposite directions, has to be marked along a line rotating through 180°, only that for the conchoid the starting-point is not the centre of rotation but the meeting-point of the moving line with a fixed line outside it. The analogy now appears in a new light. The conchoid is, so to speak, the Euclidean trace or signature of a *negative*-Euclidean circle (or of an infinite number of such circles, according to the position of the point-at-infinity O), for which the base-line or asymptote of the conchoid functions as median line, from the diametral points of which pairs of lines are to diverge keeping a *constant negative and inward distance* towards O. In the negative space determined by the circular rays (absolute involution) in O, the diametral point moves through 180° along the median line to form the total circle. (To be continued.)

GEORGE ADAMS, Clent, Stourbridge (England).

Kleine Mitteilungen

Ein Satz der elementaren Geometrie[1])

Es handelt sich um den Satz: Liegt der Punkt D nicht auf dem Umkreis des gleichseitigen Dreiecks ABC, dann lässt sich aus den Strecken $\overline{AD}$, $\overline{BD}$ und $\overline{CD}$ ein Dreieck konstruieren. Liegt D auf dem Umkreis, dann sind diese Abstände nicht Seiten eines Dreiecks, sondern lediglich Abstände von drei Punkten einer Geraden, was ohne Rechnung zu beweisen ist. Dreht man das Dreieck ABD um den Punkt A um den Winkel 60° so, dass B nach C und D nach D^* kommt, so erhält man die Strecken $DD^* = DA$, $D^*C = DB$ und DC, welche entweder ein Dreieck bilden oder auf einer Geraden liegen. Liegen die Punkte D, D^* und C auf einer Geraden, dann liegt der Punkt D (wegen des Peripheriewinkels 60° bzw. 120°) auf dem Umkreis des Dreiecks ABC.

Die Beweisfigur kann auch als Analysisfigur für die folgende Aufgabe dienen: Gegeben sind die Abstände des Punktes D von den Ecken des gleichseitigen Dreiecks ABC, das Dreieck ist zu konstruieren. Diese Aufgabe hat zwei wesentlich verschiedene Lösungen, da die Strecke DD^* Seite von zwei gleichseitigen Dreiecken ist.

R. LAUFFER, Graz.

Anmerkung der Redaktion. Wie uns Herr A. BAGER (Hiørring) mitteilt, ist die obige Ableitung auch von J. BRATU gefunden worden.

Herr J. W. A. VAN KOL (Eindhoven) teilt uns folgenden einfachen Beweis mit: Man fälle vom Punkte D die Lote auf die Seiten BC, CA, AB des gleichseitigen Dreiecks ABC. Die Fusspunkte P, Q, R bilden im allgemeinen ein nicht ausgeartetes Dreieck. Dessen Seiten PQ, QR, RP sind wegen $PQ = DC \sin 60^0$ usw. proportional den Strecken DC, DA, DB. Aus diesen lässt sich somit ein dem Dreieck PQR ähnliches Dreieck konstruieren. Liegt D auf dem Umkreis von ABC, so ergibt sich der Sonderfall, dass P, Q, R einer Geraden angehören.

Mit der nachfolgenden Arbeit schliessen wir die Publikationen zum Satz von POMPEÏU ab.

Généralisation d'un théorème de M. Pompeïu

Dernièrement, M. PAVLOVIĆ a donné dans cette revue[2]) une démonstration géométrique d'un théorème de M. POMPEÏU: On peut toujours construire un triangle avec les distances d'un point aux sommets d'un triangle équilatéral[3]). Nous donnons ici une généralisation de ce théorème pour un espace de dimension quelconque.

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[1]) Siehe unter dem gleichen Titel El. Math. *8*, 65 (1953).

[2]) S. V. PAVLOVIĆ, El. Math. *8*, 13 (1953).

[3]) D. POMPEÏU, Bull. Math. Phys. Ecole polyt. Bucarest *6*, 6–7 (1936).