Rotors of variable regular polygons

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1. Introduction

It is well known that every closed curve has a circumscribed square [1, p. 86][1]. Furthermore, for every curve of constant width, all of the circumscribed rectangles are squares. But the curves of constant width do not exhaust the class of curves all of whose circumscribed rectangles are squares. It has been mentioned recently that other curves also have this property. See [3] and [4, p. 97]. This note proposes to extend this observation to other polygons.

2. Minkowski Sum of Two Curves

Take any point as origin in a plane in which two curves are given. Let t_1 be a supporting line of one curve and let t_2, parallel to t_1, be a supporting line of the second curve. Let p_1 and p_2 be the perpendicular distances from the origin to these lines. Let t_3 be a third parallel line whose distance p_3 from the origin is equal to the sum of p_1 and p_2. Then the shape of the envelope of all lines, such as t_3, is called the Minkowski sum of the two given curves. The location of this derived curve depends upon the choice of the origin, but the shape of this curve is independent of the choice of origin. See [2, pp. 23–30] and [4, pp. 39–50].

3. Circumscribed Equiangular n-Gons

KÁRTESZI [3] noted that a closed convex curve, made of four congruent arcs (straight portions may be included) joining the vertices of a square, has the property that every circumscribed rectangle is a square. The Minkowski sums of these curves with ovals of constant width also have this property.

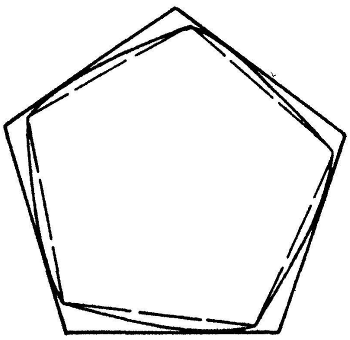
These results can be extended to other polygons. Rotors in regular polygons are curves which can be rotated through all orientations while remaining in contact with all the sides of the regular n-gon [5]. They include curves of constant width. A new curve is obtained as the Minkowski sum of such a rotor with a closed convex curve made of n congruent arcs joining the vertices of a regular n-gon. The arcs need not be symmetric, but they must be coincident by appropriate rotations confined to the plane. This new curve has the property that every circumscribed equiangular n-gon

[1]) Numbers in brackets refer to References, page 27.

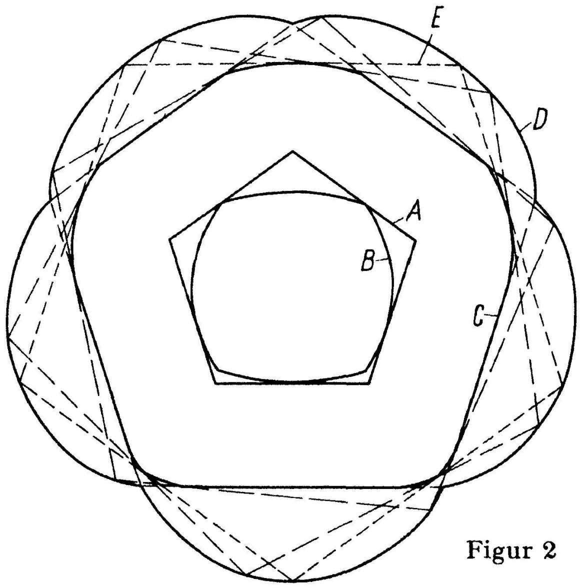
(for the given n) is also regular. This follows from the fact that the Minkowski sum of two parallel similar polygons is a third parallel similar polygon.

4. *Examples*

Since all equiangular triangles are also regular, the infinite series of new cases begins with the rotors in regular pentagons. An equiangular pentagon circumscribed about a given regular pentagon (or any closed convex curve made of five congruent arcs joining the vertices of a regular pentagon) is also regular; that is, the sides are equal as well as the angles being equal. See Figure 1. The Minkowski sum of the regular pentagon and a rotor in a regular pentagon is a curve whose equiangular circumscribed pentagons are also regular. Curve C in Figure 2 is the Minkowski sum of the inner pentagon A and its internal rotor B. Each outer pentagon E is a circumscribed equiangular pentagon of curve C; hence it is also regular.



Figur 1



Figur 2

5. *Internal Rotors of a Variable Regular n-Gon*

Select E, one of the circumscribed pentagons of the curve C. Select any arbitrary point within C as a center of rotation for the curve C. Let the five sides of the pentagon E be allowed to move independently of each other, yet constrained to remain parallel to their original positions. Then, as the curve C is rotated, the circumscribed pentagon varies in size, but it always remains regular. Hence, the curve C is called an internal rotor of the variable pentagon. In general, such internal rotors of variable regular n-gons include, as special cases, internal rotors of fixed n-gons (that is, fixed in both size and direction).

6. *External Rotors of a Variable Regular n-Gon*

If a curve of constant width is held fixed, a circumscribed square can be rotated around it. All the vertices of the square describe the same curve. This curve, therefore, has a continuous infinity of inscribed squares of the same size.

Similarly, all the vertices of a regular n-gon describe the same curve when the n-gon is rotated about one of its internal rotors. As in the case of the square, this curve has a continuous infinity of inscribed n-gons of the same size.

A regular n-curve, which consists of n congruent segments joining the vertices of a regular n-gon, has a continuous infinity of inscribed regular n-gons. If we construct a curve C, which is the Minkowski sum of a regular n-curve A and a rotor D in a regular n-gon, then all the vertices of the circumscribed regular n-gons of this curve C describe a new curve D. This curve D has an infinity of inscribed regular n-gons which are not all of the same size.

In all of the foregoing cases, the n-gon may be considered as fixed in direction, but the size may be variable as well as fixed. The new curve D may be considered as an external rotor of the variable (or constant) n-gon.

7. *Analytical Description of the Rotors*

MEISSNER [5] [6] showed that the internal rotors in a regular n-gon may be expressed by the polar tangential equation

$$p(\theta) = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\,\theta + b_k \sin k\,\theta)\,, \tag{1}$$

where $p(\theta)$ is the distance from the origin to a tangent of inclination θ and $a_k, b_k = 0$ when $k \not\equiv \pm 1 \pmod n$.

The polar tangential equation of a regular n-curve may be represented by (1) where $a_k, b_k = 0$ when $k \not\equiv 0 \pmod n$. Hence, the Minkowski sum of a rotor in an n-gon and a regular n-curve is represented by (1) where $a_k, b_k = 0$ when $k \not\equiv 0, \pm 1 \pmod n$.

The external rotors are the isoptic curves of the internal rotors and they are obtained as the locus of the vertex of an angle, equal to $\pi\,(1 - 2/n)$, whose sides are supporting lines of the internal rotor. The parametric equations of the external rotor may be obtained as follows. Let $a = p(\theta)$, and $b = p\,(\theta + \gamma)$, where $\gamma = 2\,\pi/n$. Let r be the distance from the origin to the point (x, y) on the locus. Then,

$$\gamma = \operatorname{arc\,cos}\frac{a}{r} + \operatorname{arc\,cos}\frac{b}{r} = \alpha + \beta\,.$$

Let $\lambda = \cos\gamma = a\,b/r^2 - \left(\sqrt{r^2 - a^2}\,\sqrt{r^2 - b^2}\right)/r^2$. Then,

$$\lambda\, r^2 = a\,b - \sqrt{r^4 - a^2\,r^2 - b^2\,r^2 + a^2\,b^2}\,,$$

from which $r^2 = (a^2 + b^2 - 2\,\lambda\,a\,b)/(1 - \lambda^2)$, and the sought parametric equations are the following:

$$x = a\cos\theta - \sqrt{r^2 - a^2}\sin\theta\,, \qquad y = a\sin\theta + \sqrt{r^2 - a^2}\cos\theta\,.$$

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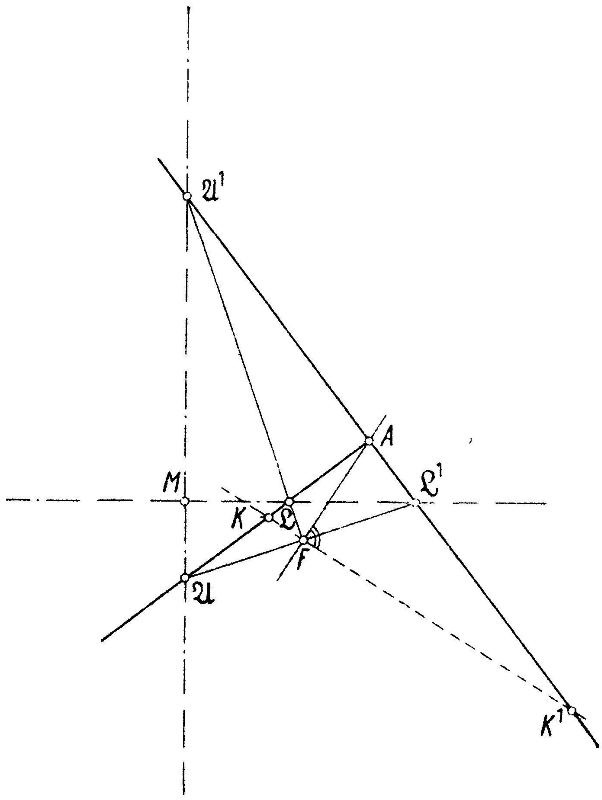
Über Kegelschnitte mit gemeinsamem Krümmungselement und Erzeugung von Steinerzykloiden

F. LAURENTI [3] [4] hat in zwei Untersuchungen gezeigt, dass die Achsen von Parabeln mit gemeinsamem Krümmungselement Steinerzykloiden als Hüllkurven besitzen. Dem analytischen Beweis von LAURENTI hat W. KICKINGER [2] eine synthetische Beweisführung gegenübergestellt, weiter lässt sich nachweisen [1], dass Ellipsen und Hyperbeln von konstantem Achsenverhältnis mit gemeinsamem Krümmungselement Steinerzykloiden als Hüllkurven ihrer Achsen besitzen. Diese letzte Aussage soll in der vorliegenden Untersuchung mit synthetischen Methoden bewiesen werden.

F. STEINER ([5], S. 205) hat folgende Konstruktion des Krümmungsmittelpunktes bei Kegelschnitten angegeben:

Treffen Tangente und Normale eines Punktes A eines Kegelschnitts die eine Achse desselben in $\mathfrak{A}$ und $\mathfrak{A}^1$, die andere in $\mathfrak{L}$ und $\mathfrak{L}^1$, errichtet man in dem Schnittpunkt $(\mathfrak{A}\,\mathfrak{L}^1, \mathfrak{A}^1\,\mathfrak{L}) = F$ auf der Geraden $A\,F$ die Senkrechte, so trifft dieselbe die Normale des Kegelschnitts in dem Krümmungsmittelpunkt.



Figur 1