Zeitschrift: Elemente der Mathematik

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 24 (1969)

Heft: 3

Artikel: Line-coloring of signed graphs

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DOI: https://doi.org/10.5169/seals-26645

ELEMENTE DER MATHEMATIK

Revue de mathématiques élémentaires – Rivista di matematica elementare

Zeitschrift zur Pflege der Mathematik
und zur Förderung des mathematisch-physikalischen Unterrichts

Publiziert mit Unterstützung des Schweizerischen Nationalfonds
zur Förderung der wissenschaftlichen Forschung

El. Math. Band 24 Heft 3 Seiten 49–72 10. Mai 1969

Line-Coloring of Signed Graphs[1])

Introduction

A *signed graph* or *sigraph* is a graph in which some of the lines have been designated as positive and the remaining as negative. Sigraphs have been studied extensively by CARTWRIGHT and HARARY (see [2] and [5]) in their theory of balance. When drawing a sigraph it is customary to indicate positive lines by solid lines and negative lines by dashed lines. Thus, the sigraph S of Figure 1 has 3 positive and 2 negative lines.

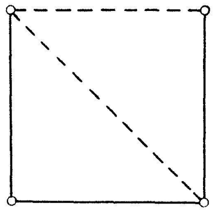

Figure 1

CARTWRIGHT and HARARY [3] have defined a sigraph S to be *colorable* if it is possible to assign colors to the points of S so that two points joined by a negative line are colored differently while two points joined by a positive line are colored the same. It was shown in [3] that a sigraph is colorable if and only if it contains no cycle with exactly one negative line. It is the purpose of this paper to define and study line-colorable sigraphs and present some of their properties. In particular, we give a characterization of line-colorable sigraphs and determine the 'line chromatic number' of special classes of sigraphs.

The *chromatic number* $\chi(S)$ of a colorable sigraph S is the smallest number of colors needed in a coloring of S. If one were to regard an ordinary graph G as a sigraph S all of whose lines are negative, then $\chi(G) = \chi(S)$. Indeed, if S is a complete colorable sigraph, then the ordinary graph G obtained by converting all negative lines to ordinary lines and deleting all positive lines has the same chromatic number as S. Thus, in a certain sense, complete colorable sigraphs and ordinary graphs are related, where negative lines correspond to ordinary lines and positive lines correspond to 'no lines'.

The *line-graph* $L(G)$ of a graph G is that graph whose points can be put in one-to-one correspondence with the lines of G so that two points of $L(G)$ are adjacent if and only if the corresponding lines of G are adjacent. In order to propose a natural defini-

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[1]) All definitions not given in this article may be found in the books [4, 5].

tion of the 'line-sigraph' of a sigraph, we again consider a complete sigraph S. Certainly, there must be a one-to-one correspondence between the points of $L(S)$ and the lines of S. Since there is a strong resemblance between the negative lines of a sigraph and the lines of an ordinary graph, the sigraph R of S induced by its negative lines should have only negative lines in its line-sigraph, while all other lines in $L(S)$ should be positive. We are thus led to the following definition. The *line-sigraph* $L(S)$ of a sigraph S is that sigraph whose points can be put in one-to-one correspondence with the lines of S in such a way that two points of $L(S)$ are joined by a negative line if and only if they correspond to two adjacent negative lines of S and are joined by a positive line if they correspond to some other two adjacent lines of S.

Since coloring the lines of an ordinary graph is equivalent to coloring the points of its line-graph, it seems natural to make the following definition. A sigraph S is *line-colorable* if its line-sigraph $L(S)$ is colorable, i.e., if it is possible to assign colors to the lines of S so that two adjacent negative lines are colored differently and any other adjacent lines are colored the same.

A Characterization of Line-Colorable Sigraphs

If v is a point of a sigraph S, then the *positive degree* $\deg^+ v$ of v is the number of positive lines of S incident with v. The *negative degree* $\deg^- v$ of v is defined analogously. We can now present the principal result of this section.

Theorem 1. *A sigraph S is line-colorable if and only if the following two properties are satisfied:*

(P1) *There exists no point v of S with $\deg^+ v \geq 1$ and $\deg^- v \geq 2$,*

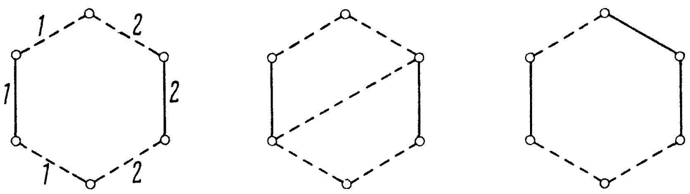
(P2) *there exists no cycle having exactly two consecutive negative linies.*

Proof. We first show the necessity of (P1) and (P2). If a point v of S is incident with one positive line and two negative lines, then these 3 lines induce a triangle in $L(S)$ having exactly one negative line so that $L(S)$ is not colorable and S is not line-colorable. Similarly, if S contains a cycle C having exactly two consecutive negative lines, then the lines of C generate a cycle in $L(S)$ having exactly one negative line, so, again, S is not line-colorable.

To prove the sufficiency of (P1) and (P2), we employ induction on the number of positive lines in a sigraph. If S has no positive lines, then S is certainly line-colorable. Assume that every sigraph having n positive lines, $n \geq 0$, and satisfying (P1) and (P2) is line-colorable. Let S be a sigraph with $n + 1$ positive lines having properties (P1) and (P2). The removal of a positive line $x = u\,v$ from S results in a sigraph S' having n positive lines. Since S' obviously satisfies (P1) and (P2), S' is line-colorable by the inductive hypothesis.

Assume that x is a bridge. If there are no lines other than x incident with u or v, then x may be colored arbitrarily in S. Otherwise, if necessary, the colors used for the component in S' containing u may be easily changed or permuted so that all lines incident with u are colored the same as those incident with v. Hence, x may be given that color thereby showing that S is line-colorable.

Suppose, on the other hand, that x is not a bridge. Then x belongs to a cycle C whose line-sequence is $x, x_1, x_2, \ldots, x_n = x$. If, in a line-coloring of S', the colors of x_1 and x_{n-1} are the same, say α, implying that all lines incident with u or v have color α, then x may be replaced and colored α also. If x_1 and x_{n-1} are colored differently, then there must exist at least 2 consecutive negative lines in C. Thus, let i be the least integer such that x_i and x_{i+1} are negative, and let j be the largest integer such that x_{j-1} and x_j are negative. By (P2), x_i and x_j are not adjacent. Let β be a color not used in coloring S', and let α_k, $k = i, j$, be the color of x_k. Also, let W_k be the set consisting of x_k and all lines colored α_k which lie on a common path with x_k. No negative line of W_i is adjacent to a negative line of W_j, for, otherwise, there would exist a cycle with exactly two consecutive negative lines, contradicting (P2). Now if the colors of the lines in $W_i \cup W_j$ are changed to β, then by replacing x and coloring it β, we have a line-coloring for S.



S_1 S_2 S_3

Figure 2

In Figure 2, S_1 is line-colorable and can be line-colored as indicated, S_2 is not line-colorable since (P1) does not hold, while S_3 is not line-colorable since (P2) does not hold.

The Line-Chromatic Number of a Sigraph

The *line-chromatic number* $\chi'(S)$ of a line-colorable sigraph S is the minimum number of colors required in a line-coloring of S. Clearly, $\chi'(S) = \chi(L(S))$.

Now we present formulas for special classes of line-colorable sigraphs, beginning with trees. Since a tree contains no cycles, by (P1) a tree is line-colorable if and only if it has no point v with $\deg^+ v \geq 1$ and $\deg^- v \geq 2$.

Theorem 2. *For any line-colorable signed tree T, $\chi'(T) = \max \deg^- v$ if T has negative lines and $\chi'(T) = 1$ otherwise.*

The proof of this theorem is straightforward and will be omitted.

A *complete sigraph* S_p has every pair of its points joined by either a positive or negative line. For $p \geq 2$, S_p is obviously line-colorable if it has no adjacent negative lines, in which case $\chi'(S_p) = 1$. Should S_p possess adjacent negative lines, then in order to satisfy (P1), there must be a point incident only with negative lines, but then to satisfy (P2) in addition, all lines must be negative. However, in this case, as we have seen, $\chi'(S_p)$ has the same value as the line-chromatic number of the ordinary complete graph K_p, which is $2\{p/2\} - 1$, as noted in [1]. We summarize this below.

Theorem 3. *Let S_p be a line-colorable complete sigraph with $p \geqq 2$ points. Then*

$$\chi'(S_p) = \begin{cases} 1 \text{ if } S_p \text{ has no adjacent negative lines.} \\ 2\{p/2\} - 1 \text{ if } S_p \text{ is all-negative.} \end{cases}$$

We now investigate *complete bipartite sigraphs* or complete sibigraphs $S_{m,n}$ whose point set V, where $|V| = m + n$, can be partitioned into subsets V_1 and V_2, with $|V_1| = m$ and $|V_2| = n$, such that every point of V_1 is joined to a point of V_2 by either a positive or negative line but no two points of the same subset V_i are adjacent.

In order to determine which of the sigraphs $S_{m,n}$ are line-colorable, we first consider the case $m \geq n \geq 3$. Again, if no two negative lines are adjacent, $S_{m,n}$ is line-colorable, and, in fact, $\chi'(S_{m,n}) = 1$. Otherwise, $S_{m,n}$ has adjacent negative lines and in order to be line-colorable and thereby satisfy (P1), it must have a point u_1 incident only with negative lines. If all other lines were positive, then there would exist a cycle (for example, $u_1 v_1 u_2 v_2 u_1$; see Figure 3a) having exactly two consecutive negative lines. Hence, $S_{m,n}$ must have at least one more negative line, say at v_1, but then all lines at v_1 are negative (see Figure 3b). However, if all lines at u_1 and v_1 are negative, then $S_{m,n}$ is all-negative, for otherwise any positive line $u_i v_j$ implies the existence of another positive line $u_i v_k$, which would produce the cycle $u_1 v_j u_i v_k u_1$ having exactly two consecutive negative lines. Therefore, if $S_{m,n}$, $m \geq n \geq 3$, is to be line-colorable and have adjacent negative lines, it has only negative lines. In this case, $\chi'(S_{m,n}) =$ max (m, n) (see König [6], p.171).

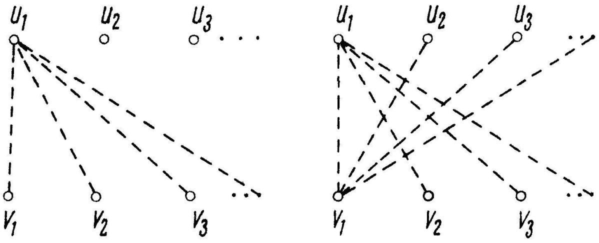


Figure 3

For $S_{m,2}$, $m \geq 3$ and $S_{m,1}$, $m \geq 1$, the situation can be handled similarly to $S_{m,n}$, $m \geq n \geq 3$, and identical results are obtained. This leaves the sigraph $S_{2,2}$ to consider. If $S_{2,2}$ contains adjacent negative lines but not all negative lines, then the only line-colorable sigraph has 3 negative lines in which case its line-chromatic number is easily seen to be 2. These results are stated in the following theorem.

Theorem 4. *A complete sibigraph $S_{m,n}$ is line-colorable if and only if*

(1) *it has no two adjacent negative lines,*

(2) *it has only negative lines, or*

(3) $m = n = 2$ *and it has* 3 *negatives lines.*

If $S_{m,n}$ is all-positive, then $\chi'(S_{m,n}) = 1$, while if $S_{m,n}$ is line-colorable but not all-positive, then $\chi'(S_{m,n})$ is the maximum negative degree.

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[2]) Research supported by grants from the U.S. Air Force Office of Scientific Research and the National Institute of Mental Health, grant MH-10834.