Randomly traversable graphs

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5.1. Die Fünfecke von Π_1 bestehen aus Würfelkanten in den Seitenflächen von Π, daher ist $\overline{\Pi}_1 = \Pi$. Auch $\overline{\Pi}_3$ ist ein Dodekaeder, denn ε_{31} ist Verbindungsebene von Würfelkanten in 5 Schnittpunkten III und ε_{31} ist zur Seitenfläche 01234 von Π parallel. $\overline{\Pi}_3$ geht aus Π durch $(\sqrt{5} - 2)$-fache Streckung aus M hervor.

5.2. In allen 9 in Figur 1 enthaltenen Schnittpunkten I', II', III' von Tetraederkanten ist deren Verbindungsebene die Ebene $\bar{3}\bar{2}6\bar{8}\bar{7}9$. Diese Ebene ist zugleich die Ebene ε_{22} von Figur 2b, sie enthält die Würfelkanten $\bar{3}9$, $\bar{7}\bar{8}$, $6\bar{2}$, die sich in 3 Ecken II von Π_2 schneiden. Daher sind $\overline{\Pi}_2$, $\overline{\Pi}'_1$, $\overline{\Pi}'_2$, $\overline{\Pi}'_3$ ein und dasselbe Ikosaeder (Mittelpunkt M, Inkugelradius = Inkugelradius der Tetraeder $= d/2\sqrt{3}$). *Dieses Ikosaeder ist die Durchschnittsmenge der zehn Tetraeder im Dodekaeder.*

5.3. Die Ebenen von $\overline{\Pi}'_4$ bilden die Seitenflächen der 5 Würfel. $\overline{\Pi}'_4$ entsteht aus Π'_4 durch Polarisieren an der Inkugel der 5 Würfel, ist also ein Rhombentriakontaeder (Mittelpunkt M, Inkugelradius $= d/2$). *Dieses Rhombentriakontaeder ist die Durchschnittsmenge der fünf Würfel im Dodekaeder.*

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Randomly Traversable Graphs

1. Introduction

A graph G is *eulerian* if it possesses a circuit containing all vertices and edges of G. These graphs are named for LEONHARD EULER [1], who encountered them while giving a solution to the Königsberg Bridge Problem. It is well known that a graph is eulerian if and only if it is connected and each of its vertices is even.

Similar to the eulerian graphs are the traversable graphs. A graph G is *traversable* if it possesses an open trail containing all vertices and edges of G. Traversable graphs are characterized (see [2], p. 65) by the properties of being connected and containing exactly two odd vertices. It is an elementary fact that every graph has an even number of odd vertices. A connected graph G with odd vertices is called *n-traversable* if there exist n open trails but no fewer which partition the edge set of G. Hence the 1-traversable graphs and the traversable graphs are identical. It follows (see [2], p. 65) that a connected graph G is n-traversable, $n \geqslant 1$, if and only if G has exactly $2n$ odd vertices.

In [3] ORE introduced an interesting class of eulerian graphs. An eulerian graph G is *randomly eulerian from a vertex v* of G if the following procedure always results in an eulerian circuit of G: Begin a trail at v by choosing any edge incident with v. Next (and at each step thereafter), the trail is continued by selecting any edge not already chosen which is adjacent with the edge most recently selected. The process terminates when no such edge is available. Equivalently, a graph G is randomly eulerian from v if every trail of G beginning at v can be extended to an eulerian circuit of G.

It is the object of this paper to study eulerian graphs which are randomly eulerian from one or more of their vertices and to extend this concept to traversable graphs and to n-traversable graphs in general.

2. Fundamental Terminology

In order to make this article self-contained, we present here those fundamental definitions which are most pertinent to our discussion. For basic graph theory terminology we follow [2].

For vertices u and v of a graph G, a *u-v trail* of G is an alternating sequence

$$u = u_1, e_1, u_2, e_2, u_3, \ldots, u_{n-1}, e_{n-1}, u_n = v \tag{1}$$

of vertices and edges of G, beginning with u and ending with v, such that each edge is incident with the two distinct vertices immediately preceding and following it and such that no edge is repeated. It should be noted that while no edge may be repeated in a trail, vertices may be repeated. Further, we may represent the trail (1) more simply as

$$u = u_1, u_2, u_3, \ldots, u_{n-1}, u_n = v, \tag{2}$$

since the edges of the trail are then evident. In general, we assume that every trail contains at least one edge and, therefore, at least two vertices. A *u-v path*, $u \neq v$, is a u-v trail in which no vertices are repeated.

A graph G is *connected* if for every two distinct vertices u and v of G, there exists a u-v trail (or u-v path) in G. A maximal connected subgraph of a graph G is called a *component* of G.

A u-v trail is *closed* if $u = v$; otherwise, it is *open*. A closed trail is also referred to as a *circuit*. A circuit in which no vertex is repeated is called a *cycle*.

A circuit containing all edges of a connected graph G is an *eulerian circuit* of G, while an open trail containing all edges of G is an *eulerian trail* of G.

Finally, the *degree* of a vertex v in a graph G, denoted $\deg v$, is the number of edges in G incident with v; the vertex v is *even* or *odd* depending on whether $\deg v$ is even or odd.

3. Randomly Eulerian Graphs

We have already noted that an eulerian graph G is randomly eulerian from a vertex v of G if and only if every trail beginning at v can be extended to an eulerian circuit of G. In Figure 1 are shown four eulerian graphs, each of which has six vertices. The graph G_0 is randomly eulerian from no vertices, G_1 is randomly eulerian from exactly one vertex, namely u, G_2 is randomly eulerian from the two vertices v and w, while G_3 is randomly eulerian from each of its vertices.

ORE [3] showed that an eulerian graph G is randomly eulerian from a vertex v of G if and only if every cycle of G contains v. With the aid of this result, it is easy to verify that the graphs of Figure 1 have the indicated properties. Moreover, it follows immediately that an eulerian graph is randomly eulerian from each of its vertices if and only if it is a cycle. We now show that the graphs of Figure 1 represent all possibilities regarding the number of vertices from which an eulerian graph is randomly eulerian.

Theorem 1. Let G be an eulerian graph with p ($\geqslant 3$) vertices. Then the number of vertices from which G is randomly eulerian is 0, 1, 2 or p.

Proof. There is obviously nothing to prove if $p = 3$, so we assume $p \geqslant 4$. Suppose the result to be false so that there exists an eulerian graph H with p ($\geqslant 4$) vertices such that H is randomly eulerian from three vertices, say u, v and w, but not from all vertices. Hence H is not itself a cycle.

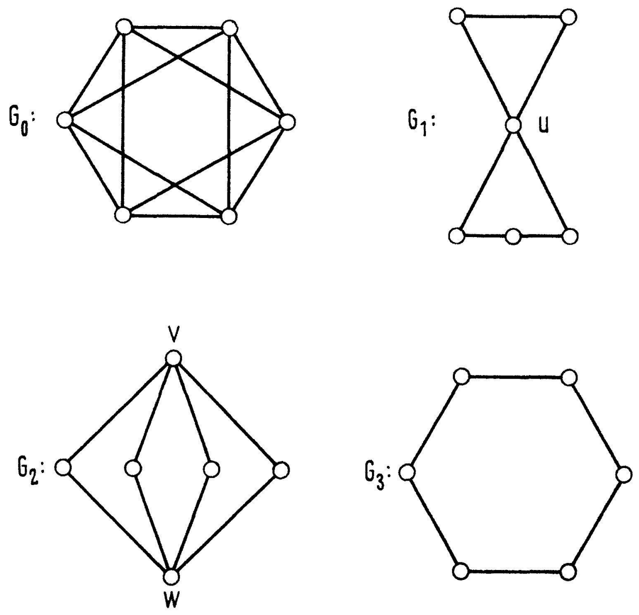

Figure 1

Since H is randomly eulerian from each of u, v and w, it follows by Ore's theorem that every cycle of H contains u, v and w. Furthermore, there exists a vertex x from which H is not randomly eulerian; therefore, not all cycles of H contain x. Let C_1 be a cycle not containing x. Because H is eulerian, there is a circuit containing x (namely an eulerian circuit) and therefore a cycle C_2 containing x. Necessarily, u, v and w also lie on C_2. Thus the distinct cycles C_1 and C_2 have at least three vertices in common.

The cycle C_2 determines two paths P_1 and P_2 connecting x with C_1. Suppose P_1 is an $x - x_1$ path while P_2 is an $x - x_2$ path, where then x_i is the only vertex of C_1 on P_i, for $i = 1, 2$; moreover, $x_1 \neq x_2$. At least one of u, v and w is neither x_1 nor x_2; suppose u is such a vertex. Hence C_1 determines two $x_1 - x_2$ paths, only one of which contains u; suppose Q is the $x_1 - x_2$ path not containing u. Hence, if we begin with P_1, follow Q, and then proceed from x_2 to x along P_2, we have a cycle not containing u, which produces a contradiction.

4. Randomly Traversable Graphs

We define a traversable graph G to be ***randomly traversable from a vertex v*** if every trail in G with initial vertex v can be extended to an eulerian trail of G. Naturally, such a vertex v is necessarily an odd vertex of G, implying that a traversable graph is randomly traversable from at most two of its vertices. Figure 2 shows traversable graphs H_0, H_1, H_2 such that H_k, $k = 0, 1, 2$, is randomly traversable from k of its vertices. A traversable graph G is said to be ***randomly traversable*** if it is randomly traversable from both of its odd vertices.

It is possible to characterize traversable graphs which are randomly traversable from a given vertex in much the same way as ORE did for randomly eulerian graphs.

Theorem 2. Let u and v be the two odd vertices of a traversable graph G. Then G is randomly traversable from u if and only if every cycle of G contains v.

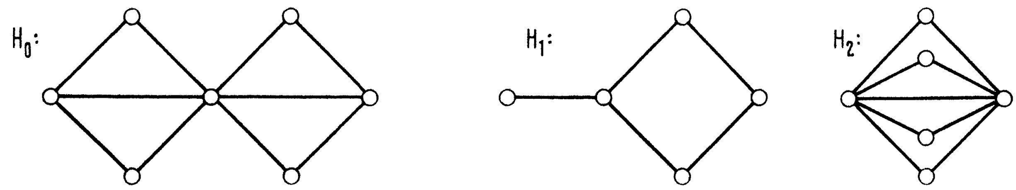

Figure 2

Proof. Suppose G is randomly traversable from u, and assume G has a cycle C not containing v. Denote by H the graph obtained by removing the edges of C from G. Necessarily, each vertex of H has the same parity as it does in G; therefore, u and v are the only two odd vertices of H and thus belong to the same component H_1 of H. Hence H_1 is traversable and has a u-v trail P_1 containing all edges of H_1. Since P_1 contains all edges of G incident with v, the trail P_1 cannot be extended to an eulerian trail of G, contradicting the fact that G is randomly traversable from u.

Conversely, suppose every cycle of G contains v, and assume G is not randomly traversable from u. Hence there exists a maximal trail P of G beginning at u which cannot be extended to an eulerian u-v trail. Thus P is a u-v trail not containing all edges of G. By deleting the edges of P from G, a nonempty graph G' results in which every vertex is even and v is isolated. There exists a nontrivial component H' of G'; thus H' is eulerian, contains an eulerian circuit, and therefore contains a cycle C. Since C does not contain v, a contradiction results.

Corollary 2a. Let u and v be the two odd vertices of a traversable graph G. Then G is randomly traversable if and only if every cycle of G contains both u and v.

5. Randomly n-Traversable Graphs

Just as n-traversable graphs constitute a generalization of traversable graphs, we now introduce the concept of randomly n-traversable graphs as a generalization of randomly traversable graphs.

For a graph G, we denote its edge set by $E(G)$. Similarly, the edge set of a trail T of a graph G is denoted $E(T)$. By $G - E(T)$ we mean the graph obtained by deleting the edges of T from G. A trail T of a graph G, having initial vertex v and terminal vertex w, is said to be *maximal from* v if every edge of G incident with w belongs to T.

An n-traversable graph G (which necessarily then has $2n$ odd vertices) is ***randomly n-traversable from an odd vertex*** v if for every sequence $v_1, v_2, \ldots, v_n$ of n odd vertices of G for which $v_1 = v$ and for every n trails $T_1, T_2, \ldots, T_n$ such that T_1 is maximal from v_1 in G and T_i is maximal from v_i in

$$G - \bigcup_{j=1}^{i-1} E(T_j)\,, \qquad i = 2, 3, \ldots, n\,,$$

it follows that $E(G) = \bigcup_{i=1}^{n} E(T_i)$. A graph is *randomly n-traversable* if it is randomly n-traversable from each of its odd vertices. We note then that the randomly 1-traversable graphs coincide with the randomly traversable graphs. The following theorem gives a necessary condition for a graph to be randomly n-traversable from one of its odd vertices.

Theorem 3. If G is a graph which is randomly n-traversable from an odd vertex v, then every cycle of G contains an odd vertex other than v.

Proof. Let C be an arbitrary cycle in G, and consider $G - E(C)$, which has $2\,n$ odd vertices. Let T_1 be a trail in $G - E(C)$ which is maximal from its initial vertex v, while for $i = 2, 3, \ldots, n$, let T_i be a trail maximal from an odd vertex v_i in $G - E(C) - \bigcup_{j=1}^{i-1} E(T_j)$. Each of these trails necessarily terminates in an odd vertex of G. Either T_1 is not maximal in G or T_i is not maximal in $G - \bigcup_{j=1}^{i-1} E(T_j)$ for some $i = 2, 3, \ldots, n$; for if all n trails are maximal in these respective graphs, then $\bigcup_{i=1}^{n} E(T_i) \neq E(G)$, which contradicts the fact that G is randomly n-traversable from v. Thus the terminal vertex of at least one trail T_i lies on C so that C contains an odd vertex of G other than v.

The necessary condition for a graph to be randomly traversable given in Theorem 2 now follows as a corollary to Theorem 3. From Theorem 3 we may now derive a necessary condition for a graph to be randomly n-traversable.

Corollary 3a. If G is a randomly n-traversable graph, then every cycle of G contains at least two odd vertices.

Proof. Let C be a cycle of G. By Theorem 3, C contains at least one odd vertex, say u. By hypothesis, G is randomly n-traversable from u so that, again by Theorem 3, C contains an odd vertex other than u, completing the proof.

Next we present a sufficient condition for an n-traversable graph to be randomly n-traversable from one of its odd vertices.

Theorem 4. Let G be an n-traversable graph with odd vertex v. If every cycle of G contains at least n odd vertices other than v, then G is randomly n-traversable from v.

Proof. Let $v_1, v_2, \ldots, v_n$ be n odd vertices of G, where $v_1 = v$, and let $T_1, T_2, \ldots, T_n$ be n trails so that T_1 is maximal from v_1 in G and T_i is maximal from v_i in

$$G - \bigcup_{j=1}^{i-1} E(T_j)\,, \qquad i = 2, 3, \ldots, n\,.$$

Since T_i, $i = 1, 2, \ldots, n$, is a trail which is maximal from v_i, it must terminate at a vertex w_i having degree zero in

$$H_i = G - \bigcup_{j=1}^{i} E(T_j)\,.$$

Because every vertex which is even in G is also even in H_{i-1}, w_i is necessarily odd in G. Furthermore, H_i has exactly $2\,n - 2\,i$ odd vertices. Hence H_n has only even vertices. If H_n has no edges, then $E(G) = \bigcup_{j=1}^{i} E(T_i)$, which produces the desired result. Suppose, then, that H_n has edges. In this case, H_n contains cycles; thus let C be a cycle in H_n. By hypothesis, C contains at least n odd vertices of G other than v. Since G has exactly $2n$ odd vertices, C must contain a vertex w_k, $1 \leqslant k \leqslant n$. However, w_k has degree zero in H_k as well as in H_n. This produces a contradiction, completing the proof.

The sufficient condition for a graph to be randomly traversable given in Theorem 2 now follows as a corollary to Theorem 4.

The converse of the preceding theorem does not hold, in general. For example, the 2-traversable graph G of Figure 3 is randomly 2-traversable from v; however, the only cycle of G contains only one odd vertex.

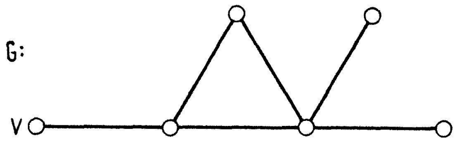

Figure 3

Corollary 4a. If every cycle of an n-traversable graph G contains at least $n + 1$ odd vertices, then G is randomly n-traversable.

The number of odd vertices in the statement of Corollary 4a cannot be reduced, as the following example shows. Let G be the graph consisting of a cycle C: $v_1, v_2, \ldots, v_n, v_1$, n additional vertices $u_1, u_2, \ldots, u_n$, and the edges $u_i v_i$, $i = 1, 2, \ldots, n$. Figure 4 illustrates the graph G for the case $n = 5$. Although G is n-traversable, and the only cycle of G contains exactly n odd vertices, G is not randomly n-traversable. For example, the n trails v_i, u_i, $i = 1, 2, \ldots, n$ do not partition the edge set of G.

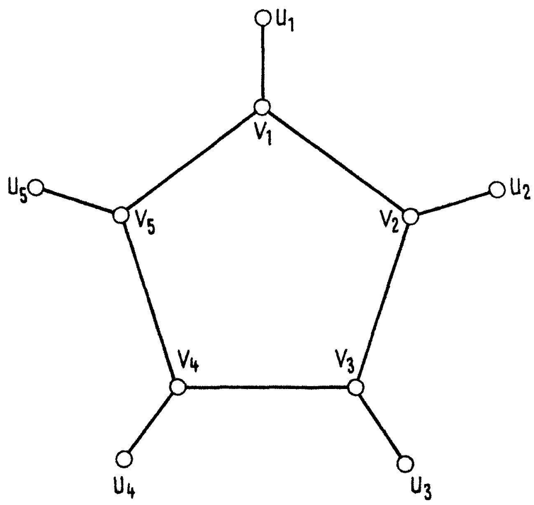

Figure 4

We conclude by verifying the converse of Corollary 4a for the case $n = 2$.

Theorem 5. If G is a randomly 2-traversable graph, then every cycle of G contains at least three odd vertices.

Proof. By Corollary 3a, every cycle of G contains at least two odd vertices. Suppose there exists a cycle C in G containing exactly two odd vertices, say u and v, and let u_1 and v_1 be the remaining odd vertices of G. We consider two cases.

Case 1. There exists a $u - u_1$ path in $G - E(C)$ not containing v_1 or a $u - v_1$ path in $G - E(C)$ not containing u_1. Without loss of generality, we assume the former, denoting the path by P. The graph $G - E(C) - E(P)$ has exactly two odd vertices, namely v and v_1, which necessarily belong to the same component G' of $G - E(C) - E(P)$. Furthermore, the degree of v_1 is the same in $G - E(C) - E(P)$ as in G. Let T be a v-v_1 eulerian trail in G'; the trail T is therefore maximal from v in G. Let T_1 be a maximal trail from u in $G - E(C) - E(T)$, necessarily terminating at u_1. Then T_1 is also maximal in $G - E(T)$. However, $E(T) \cup E(T_1) \neq E(G)$, which is contradictory.

Case 2. There exists no $u - u_1$ path in $G - E(C)$. If there exists a $u - v_1$ path in $G - E(C)$, then we are in Case 1 and a contradiction results. Hence we may assume that $G - E(C)$ has a u-v path P containing neither u_1 nor v_1. If P has a vertex of C different from u or v, then G has a cycle containing only one odd vertex, namely u, which is impossible. Now the cycle C determines two edge-disjoint u-v paths P_1 and P_2. Since G is connected, there exists in G either a $u - u_1$ path not containing v or a $v - u_1$ path not containing u; assume the former, denoting the $u - u_1$ path by P_3. We further suppose that P_3 does not contain v_1; otherwise, we let P_3 denote the resulting $u - v_1$ path. The path P_3 has at least one edge which is also an edge of C; furthermore, P_3 contains vertices of only one of P_1 and P_2 (with the exception of the vertex u), for otherwise G has a cycle containing only one odd vertex. Suppose P_1 contains a vertex of P_3 different from u or v so that P_2 has no such vertex. The $u - v$ paths P and P_2 combine to form a cycle C_1 containing u and v but neither u_1 nor v_1. However, a $u - u_1$ path exists in $G - E(C_1)$ returning us to Case 1 which yields a contradiction and completes the proof.

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Über hebbare Unstetigkeiten

Die vorliegende Note ist als Beitrag zur Sammlung pathologischer Beispiele der Analysis gedacht, wie sie etwa in [1] gegeben wird.

Wir betrachten die Menge $\mathfrak{J}[a, b]$ der auf dem abgeschlossenen Intervall $[a, b]$ definierten Funktionen, die in jedem Punkt von $[a, b]$ unstetig sind. Eine solche Funktion ist beispielsweise

$$f(x) = \begin{cases} +1 & \text{für } x \in \mathbf{Q} \\ -1 & \text{für } x \in \mathbf{R} - \mathbf{Q}, \end{cases}$$