

# Note on a diophantine equation

Autor(en): **Williams, H.C.**

Objektyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **25 (1970)**

Heft 6

PDF erstellt am: **08.08.2024**

Persistenter Link: <https://doi.org/10.5169/seals-27360>

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a distance. Can this be done with as few as  $O(n^{1/2})$  points; or with  $O(n^{1/3})$  points in one dimension?

Another open problem [1] is given any  $n$  points in the plane (not necessarily lattice points) [or in  $d$  dimensions], how many can one select so that the distances which are determined are all distinct? P. ERDÖS and R. K. GUY, Budapest

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### Note on a Diophantine Equation

SCHINZEL and SIERPIŃSKI [1] have given the general solution of the diophantine equation

$$(x^2 - 1)(y^2 - 1) = \left[ \left( \frac{x - y}{2} \right)^2 - 1 \right]^2,$$

and SZYMICZEK [2] has given the general solution of

$$(x^2 - z^2)(y^2 - z^2) = \left[ \left( \frac{y - x}{2} \right)^2 - z^2 \right]^2.$$

The purpose of this paper is to obtain a complete solution of the diophantine equation

$$(x^2 + a)(y^2 + a) = \left[ a \left( \frac{y - x}{2b} \right)^2 + b^2 \right]^2, \tag{1}$$

where  $a$  and  $b$  are any two given integers.

Let  $X = x - y$ ,  $Y = x + y$ ; then  $X \equiv Y \pmod{2}$  and (1) becomes

$$b^4 (X^2 + 2XY + Y^2 + 4a)(X^2 - 2XY + Y^2 + 4a) = (aX^2 + 4b^4)^2.$$

This equation reduces to

$$b^4 ((Y^2 - X^2)^2 + 8a(Y^2 - X^2) + 16a^2) = (aX^2 + 4b^4)^2 - 16ab^4X^2$$

and we have

$$b^2 (Y^2 - X^2 + 4a) = \pm (aX^2 - 4b^4).$$

At this point it becomes necessary to find the complete solutions of

$$b^2 (Y^2 - X^2 + 4a) = + (aX^2 - 4b^4) \quad (2)$$

and

$$b^2 (Y^2 - X^2 + 4a) = - (aX^2 - 4b^4) . \quad (3)$$

We shall first solve (2).

Let  $(a, b^2) = \gamma^2 \delta$ , where  $\delta$  has no square factors; then  $a = \gamma^2 \delta \alpha$  and  $b = \gamma \delta \beta$ , where  $(\beta \delta, \alpha) = 1$ .

From (2), we must have

$$\delta \beta^2 \mid \alpha X^2$$

or, equivalently,  $X = \delta \beta Z$ , for some integer  $Z$ . On substituting this value for  $X$  and re-arranging the terms in (2), we obtain the equation

$$Y^2 - (\delta \beta^2 + \alpha) \delta Z^2 = -4 \gamma^2 (\delta^2 \beta^2 + \delta \alpha) . \quad (4)$$

If we let

$$c_1^2 d_1 = \delta \beta^2 + \alpha , \quad (5)$$

where  $d_1$  has no square factors, then  $(d_1, \delta) = 1$  and therefore,  $Y = c_1 d_1 \delta W$ , for some integer  $W$ . Equation (4) reduces to

$$Z^2 - d_1 \delta W^2 = 4 \gamma^2 . \quad (6)$$

Thus the problem of obtaining all solutions of (2) reduces to the solved problem of finding all solutions of (6). It may be shown similarly that if we let

$$c_2^2 d_2 = \delta \beta^2 - \alpha ,$$

where  $d_2$  has no square factor, then the problem of obtaining all solutions of (3) reduces to the problem of obtaining all solutions of

$$z^2 - d_2 \delta w^2 = -4 \gamma^2 .$$

It is evident that part of the solutions of (1) must be of the form

$$x = (\delta \beta Z + c_1 d_1 \delta W)/2 , \quad y = (c_1 d_1 \delta W - \delta \beta Z)/2 ,$$

where  $(Z, W)$  is a solution of (6). If  $2 \mid \delta$  or  $2 \mid d_1$ , it is clear that  $x$  and  $y$  will both be integers. If  $2 \nmid \delta$  and  $2 \nmid d_1$ , then, by (6),  $Z$  and  $W$  will be of the same parity. If they are both even, then  $x$  and  $y$  are integers; if they are both odd, we must have, in order that  $x$  and  $y$  be integers, that  $\beta \equiv c_1 \pmod{2}$ . But, from (5),

$$c_1 \equiv \beta + \alpha \pmod{2} ;$$

thus, it is necessary that  $\alpha \equiv 0 \pmod{2}$ .

Hence, if  $2 \mid (d_1 \delta \alpha)$ ,  $(c_1 d_1 \delta W \pm \delta \beta Z)/2$  are integers. If  $2 \nmid (d_1 \delta \alpha)$ , then  $(c_1 d_1 \delta W \pm \delta \beta Z)/2$  will be integers if and only if  $2 \mid W$  and  $2 \mid Z$ .

This same discussion may be applied to what must be the forms of the remaining solutions of (1), that is,

$$x = (\delta \beta z + c_2 d_2 \delta w)/2 , \quad y = (c_2 d_2 \delta w - \delta \beta z)/2 .$$

Thus we have proved the following

*Theorem.* If  $(a, b^2) = \gamma^2 \delta$ , where  $\delta$  has no square factors, and

$$c^2 d = \delta \beta^2 \pm \alpha,$$

where

$$\beta = \frac{b}{\gamma \delta},$$

and

$$\alpha = \frac{a}{\gamma^2 \delta},$$

and  $d$  has no square factors, then any solution of (1) must be of the form

$$x = (c d \delta W + \delta \beta Z)/2, \quad y = (c d \delta W - \delta \beta Z)/2,$$

where  $(Z, W)$  is a solution of

$$Z^2 - d \delta W^2 = \pm 4 \gamma^2$$

and  $2 \mid (\alpha \delta d)$ ; if  $2 \nmid (\alpha \delta d)$ , then any solution of (1) must be of the form

$$x = c d \delta W + \delta \beta Z, \quad y = c d \delta W - \delta \beta Z,$$

where  $(Z, W)$  is a solution of

$$Z^2 - d \delta W^2 = \pm \gamma^2.$$

This theorem has the following

*Corollary.* The equation

$$(x^2 + a)(y^2 + a) = (a z^2 + b^2)^2, \quad (7)$$

where  $a$  and  $b$  are integers, has integer solutions given by

$$x = b u + c d v, \quad y = b u - c d v, \quad z = u,$$

provided

$$b^2 \pm a = c^2 d,$$

where  $d$  has no square factors, and  $u$  and  $v$  are given by

$$u^2 - d v^2 = \pm 1.$$

It is not known to the author whether there are values of  $a$  and  $b$  for which the above result provides all the solutions to the equation (7); however, the equation (7) with  $a = b = 1$ , which was originally suggested to the author by L. J. MORDELL, has no solutions other than those predicted by the foregoing method for all integers  $x$  and  $y$ , where  $1 \leq x \leq 300$  and  $1 \leq y \leq x$ .

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