A new method of evaluating the sums of [formula] and related series

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Objekttyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **27 (1972)**

Heft 2

PDF erstellt am: **08.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-28628

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A New Method of Evaluating the Sums of $\sum_{k=1}^{\infty}(-1)^{k+1}k^{-2p}$, $p = 1, 2, 3, \ldots$ and Related Series

The decisive tool in our attempt to evaluate the sum $\sum_{k=1}^{\infty}(-1)^{k+1}k^{-2p}$ for fixed $p \in N = \{1, 2, 3, \ldots\}$ is the kernel of Dirichlet in both of its representations (for all real x)

$$D_n(x) = \frac{\sin(2n+1)\frac{x}{2}}{2\sin\frac{x}{2}} = \frac{1}{2} + \sum_{k=1}^{n}\cos kx \qquad (n \in N)\,. \tag{1}$$

First of all, let us consider

$$C_p(k) = \int_0^{\pi} t^{2p}\cos kt\,dt\,, \qquad k \in N;$$

a twofold integration by parts gives the recursive formula

$$C_p(k) = \frac{2p}{k^2}\{(-1)^k\pi^{2p-1} - (2p-1)\,C_{p-1}(k)\} \qquad (p \in N)$$

$$C_0(k) = 0$$

and hence, as is immediately verified[1]),

$$C_p(k) = (-1)^k\pi(2p)!\sum_{j=1}^{p}(-1)^{j+1}\frac{\pi^{2(p-j)}}{(2(p-j)+1)!}\frac{1}{k^{2j}} \qquad (p \in N)\,. \tag{2}$$

[1]) Formula 3.529.1 in [4] is obviously incorrect.

Now, concerning the even moments of (1), i.e.,

$$M(2p; n) = \int_0^{\pi} t^{2p} D_n(t)\,dt \qquad (p \in \mathsf{N}),$$

we have, on the one hand, by the polynomial representation

$$M(2p; n) = \frac{\pi^{2p+1}}{2(2p+1)} + \sum_{k=1}^{n} C_p(k); \tag{3}$$

and on the other,

$$M(2p; n) = 2^{2p} \int_0^{\pi/2} \frac{y^{2p}}{\sin y} \sin(2n+1)\,y\,dy\,.$$

The second mean-value theorem (cf. e.g. [6, p. 163]) yields

$$\int_0^{\pi/2} \frac{y^{2p}}{\sin y} \sin(2n+1)\,y\,dy = \left(\frac{\pi}{2}\right)^{2p} \frac{\cos(2n+1)\,\xi}{2n+1} \qquad \left(0 \leqslant \xi \leqslant \frac{\pi}{2}, \; n \in \mathsf{N}\right)$$

since $\sin(2n+1)\,y$ is continuous, whereas $y^{2p}/\sin y$ is non-negative and increasing for $y \in [0, \pi/2]$. Thus, uniformly for all $p \in \mathsf{N}$,

$$M(2p; n) = O(n^{-1}) \qquad (n \to \infty).$$

This gives, using (3) and (2),

$$\lim_{n\to\infty} \left\{\frac{\pi^{2p}}{2(2p+1)} - \sum_{k=1}^{n} (-1)^{k+1} (2p)! \sum_{j=1}^{p} (-1)^{j+1} \frac{\pi^{2(p-j)}}{(2(p-j)+1)!} \frac{1}{k^{2j}}\right\} = 0$$

or

$$\sum_{k=1}^{\infty} (-1)^{k+1} \sum_{j=1}^{p} (-1)^{j+1} \frac{\pi^{2(p-j)}}{(2(p-j)+1)!} \frac{1}{k^{2j}} = \frac{\pi^{2p}}{2(2p+1)!} \qquad (p \in \mathsf{N}).$$

After extracting the term for $j = p$, the desired formula reads

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2p}} = (-1)^{p+1} \frac{\pi^{2p}}{2(2p+1)!} - \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{j=1}^{p-1} (-1)^{p+j} \frac{\pi^{2(p-j)}}{(2(p-j)+1)!} \frac{1}{k^{2j}}. \tag{4}$$

From (4), by recursion for $p = 1, 2, 3, \ldots$ the sums are easily calculated: in case $p = 1$ it follows directly that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}; \tag{5}$$

for $p = 2$, together with (5),

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} = -\frac{\pi^4}{240} + \frac{\pi^2}{6} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = -\frac{\pi^4}{240} + \frac{\pi^4}{72},$$

so that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} = \frac{7}{720} \pi^4 ; \tag{6}$$

continuing in this way, for $p = 3$ now using both (5) and (6), one obtains

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^6} = \frac{31}{30240} \pi^6, \ldots .$$

The standard proof (see e.g. [2, p. 245ff], [1, p. 244]) by means of the theory of Bernoulli polynomials leads to

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2p}} = \frac{(-1)^{p+1} \pi^{2p}}{2(2p)!} (2^{2p} - 2) B_{2p} \qquad (p \in N) \tag{7}$$

where B_{2p} are the Bernoulli numbers. These, in turn, may be deduced from the symbolic equation (i.e. expand (8) and set formally $B^n = B_n$)

$$(B+1)^n - B^n = 0 , \quad B_0 = B^0 = 1 , \quad n = 1, 2, 3, \ldots , \tag{8}$$

cf. [1, p. 233], [2, p. 185].

Of course, neither (4) nor (7) are actually closed expressions for the sum since the Bernoulli numbers are obtainable only via the recurrence formula (8). But (4) has the advantage of avoiding the use of B_{2p}; moreover a comparison of (7) and (4) gives another way of evaluating these numbers.

In view of the obvious relations, cf. [2, p. 246],

$$\sum_{k=1}^{\infty} \frac{1}{k^{2p}} = \frac{2^{2p}}{2^{2p}-2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2p}} ,$$
$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2p}} = \frac{2^{2p}-1}{2^{2p}-2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2p}} \qquad (p \in N)$$

the sums of these series may also be derived immediately from (4).

In the particular case $p = 1$, there is a proof in [5] using another kernel, namely that of Fejér, whereas the method employed in [3] is implicitely based upon the kernel of de La Vallée Poussin. This reveals that these kernels originating from approximation theory, also play an interesting rôle in a quite different branch of analysis.

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[2]) The author is indebted to Professor P. L. BUTZER for many helpful remarks. A support of the *Deutsche Forschungsgemeinschaft* is greatly acknowledged.