# Hypo-eulerian and hypo-traversable graphs

Autor(en): **Kapoor, S.F.**

Objekttyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **28 (1973)**

Heft 5

PDF erstellt am: **12.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-29459

*Proof.* Let $T_0$ be a simplex of minimal volume containing $K$. By the theorem of Day [2], the centroids of the facets of $T_0$ touch $K$. Let $t$ be the simplex whose vertices are those centroids, and let $T$ be the simplex parallel to $t$ and circumscribed about $K$. Then $t = (n^{-n})\, T_0$ and $T \geq T_0$, so

$$K^n \geq t^{n-1}\, T \geq (n^{-n(n-1)}\, T_0^{n-1})\, (T_0)\,, \tag{11}$$

so $T_0 \leq (n^{n-1})\, K$, as we wanted to prove.

G. D. Chakerian, University of California, Davis

REFERENCES

[1] G. D. CHAKERIAN and L. H. LANGE, *Geometric Extremum Problems*, Math. Mag. *44*, 57–69 (1971).
[2] M. M. DAY, *Polygons Circumscribed About Closed Convex Curves*, Trans. Amer. Math. Soc. *62*, 315–319 (1947).
[3] C. H. DOWKER, *On Minimum Circumscribed Polygons*, Bull. Amer. Math. Soc. *50*, 120–122 (1944).
[4] L. FEJES TÓTH, *Lagerungen in der Ebene, auf der Kugel und im Raum* (Berlin 1953).
[5] L. A. LYUSTERNIK, *Convex Figures and Polyhedra*, Moscow 1956 (Russian; English translation, New York 1963,).

# Hypo-Eulerian and Hypo-Traversable Graphs

## Introduction

If a graph $G$ does not possess a given property $P$, and for each vertex $v$ of $G$ the graph $G - v$ enjoys property $P$, then $G$ is said to be a *hypo-P* graph. Recently, studies have been made where $P$ stands for the graph being *hamiltonian*, *planar*, and *outerplanar* (e.g., see [3]). Here we obtain a characterization of *hypo-eulerian* and *hypo-randomly-eulerian* graphs, and investigate in this respect some of the other concepts arising out of Euler's solution of the classical Königsberg Seven Bridges Problem.

## Preliminaries

Following the terminology of [2], a *graph* will be finite, undirected, without loops or multiple edges. A *walk* of a graph $G$ is an alternating sequence $v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$ of vertices and edges of $G$, beginning and ending with vertices and where the edge $e_i = v_{i-1} v_i$ for $i = 1, 2, \ldots, n$. This is a $v_0 - v_n$ walk, and is usually denoted $v_0 v_1 v_2 \ldots v_n$; it is *closed* if $v_0 = v_n$ and *open* otherwise. A walk is a *trail* if all its edges are distinct; it is a *path* if all its vertices are distinct. A closed trail is a circuit and a circuit on distinct vertices is a *cycle*. A cycle on $p$ vertices is denoted $C_p$, and $C_3$ is called a *triangle*.

If for every two distinct vertices $u$ and $v$ of a graph $G$ there exists a $u - v$ path, then $G$ is *connected*. A *component* of $G$ is a maximal connected subgraph of $G$. A vertex

$v$ is a *cutpoint* of $G$ if $G - v$ has more components than $G$. An eulerian circuit of a graph $G$ is a circuit which contains all the vertices and edges of $G$, and an *eulerian trail* of $G$ is an open trail which contains all the vertices and edges of $G$; in either case $G$ has to be connected. We will assume that an eulerian circuit or an eulerian trail has at least one edge in it.

The number of edges incident with a vertex $v$ is the *degree* of $v$ which is written as *deg* $v$. Let $\delta(G) = \min_v \deg v$ and $\Delta(G) = \max_v \deg v$. A graph $G$ is *regular of degree r* (or *r-regular*) if $\delta(G) = \Delta(G) = r$. A *cubic* graph is 3-regular. We use $p(G)$ and $q(G)$ (often simply $p$ and $q$) for the number of vertices and edges of a graph $G$. The *trivial* graph has $p = 1$ and the *complete graph* $K_p$ on $p$ vertices has $q = p\,(p-1)/2$. The *complete bipartite graph* $K(m, n)$ has its vertex set partitioned into nonempty sets $V_1$ and $V_2$ containing $m$ and $n$ elements respectively such that $uv$ is an edge of $K(m, n)$ if and only if $u \in V_i$ and $v \in V_j$, $i \neq j$.

An edge $x = u\,v$ of a graph $H$ is said to be *subdivided* if it is replaced by a new vertex $w$ together with the edges $u\,w$ and $w\,v$. A graph $G$ is *homeomorphic from* a graph $H$ if $G$ can be obtained from $H$ by a finite sequence of such subdivisions. Two graphs $G_1$ and $G_2$ are *homeomorphic* if there exists a graph $G$ such that $G_1$ and $G_2$ are both homeomorphic from $G$.

Let $\theta(G)$ ($\xi(G)$) consist of the vertices of $G$ having their degrees odd (even). Let the number of elements in $\theta(G)$ be called the *euler number* of $G$, and let this be written as $\in(G)$. Then $\in(G)$ is a nonnegative even integer.

## Hypo-eulerian Graphs

A graph $G$ on $p \geq 3$ vertices is defined to be *eulerian* if it possesses an eulerian circuit. The next result is well known.

*Theorem* (*Euler*). Let $G$ be a connected graph. Then $G$ is eulerian if and only if $\in(G) = 0$.

By definition, a graph $G$ is *hypo-eulerian* if $G$ is not eulerian, but the graph $G - v$ is eulerian for each vertex $v$ of $G$.

*Theorem* 1. Let $G$ be a connected nontrivial graph. Then $G$ is hypo-eulerian if and only if $G = K_{2n}$, $n \geq 2$.

*Proof*. Clearly, $\in(K_{2n}) = 2\,n > 0$ and $\in(K_{2n} - v) = \in(K_{2n-1}) = 0$ imply the sufficiency part. So let $G$ be a nontrivial connected hypo-eulerian graph. As $G - v$ is eulerian, $p(G) \geq 4$.

First we show that every vertex of $G$ must be odd. Assume that $\xi(G) \neq \phi$, and let $u \in \xi(G)$. Now $u$ must be adjacent with only odd vertices otherwise $\in(G - u) > 0$. On the other hand if $v \in \theta(G)$, then for the same reason $v$ must also be adjacent with only odd vertices. This contradicts $\xi(G) \neq \phi$. Hence $p(G) = \in(G) = 2\,n$ for some $n \geq 2$.

Secondly, we assert that $G$ is complete. For if not, there exist two nonadjacent odd vertices $u$ and $v$ in $G$. Now the vertex $v$ has odd degree in $G - u$ and contradicts $\in(G - u) = 0$. This completes the proof.

If $G$ is an eulerian graph with $p \geq 3$ and $v$ is any vertex of $G$, then $G - v$ necessarily contains odd vertices and must be noneulerian. This we mention next.

*Theorem* 2. Let $G$ be a connected nontrivial graph. Then $G$ is hypo-noneulerian if and only if $G$ is eulerian.

Ore [4] called an eulerian graph $G$ *randomly eulerian from a vertex* $v$ if every trail of $G$ beginning at $v$ can be extended to an eulerian circuit of $G$; a graph $G$ is *randomly eulerian* if it is randomly eulerian from each of its vertices. Ore characterized graphs which are randomly eulerian from a vertex $v$ as those graphs in which $v$ belongs to every cycle of $G$. This leads to the result that $G$ is randomly eulerian if and only if $G$ is a cycle.

*Theorem* 3. A graph $G$ is hypo-randomly-eulerian if and only if $G = K_4$.

*Proof.* Since a cycle is obtained by deleting any vertex of $K_4$, this graph certainly has the desired property. Conversely, let $G$ be a hypo-randomly-eulerian graph. Observe that in view of Theorem 2, $G$ and $G - v$ cannot be both eulerian for any vertex $v$. Hence $G$ is necessarily hypo-eulerian, and by Theorem 1, $G = K_{2n}$ for some $n \geq 2$. Moreover, since $G - v$ must be a cycle for each vertex $v$ of $K_{2n}$, we conclude that $G = K_4$.

Chartrand and White [1] proved that if $G$ is an eulerian graph which is randomly eulerian from $k$ vertices, then $k = 0, 1, 2$ or $p(G)$, and following this we will denote a graph which is randomly eulerian from $k$ vertices as an $RE(k)$ graph. A study of *hypo-$RE(k)$* graphs is now in order. Let $G$ be a graph which is not $RE(k)$, but let $G - v$ be randomly eulerian from $k$ vertices. Then, as stated earlier, $G$ must be a hypo-eulerian graph with the additional property that for all $v$, $G - v$ is an $RE(k)$ graph. So by Theorem 1, $G = K_{2n}$ and $G - v = K_{2n-1}$, $n \geq 2$. When $n \geq 3$, for every vertex $u$ of $G - v$ we can find a cycle, namely a triangle, which avoids $u$, and so $G - v$ is an $RE(o)$ graph. The case $n = 2$ yields that $G - v$ is an $RE(p)$ graph. Also, $G - v$ is not an $RE(k)$ graph for $k = 1$ and $k = 2$. These remarks lead to the next result where we note that the hypo-$RE(p)$ graphs have already been described in Theorem 3.

*Theorem* 4.

(a) A graph $G$ on $p \geq 4$ vertices is hypo-$RE(o)$ if and only if $G = K_{2n}$, $n \geq 3$.
(b) No graph is hypo-$RE(1)$ or hypo-$RE(2)$.
(c) A graph $G$ on $p \geq 4$ vertices is hypo-$RE(p)$ if and only if $G = K_4$.

We conclude this section by stating a result analogous to Theorem 2.

*Theorem* 5. A graph $G$ is hypo-non$RE(k)$ if and only if $G$ is an $RE(k)$ graph.

## Hypo-traversable Graphs

A graph $G$ on $p \geq 2$ vertices is said to be *traversable* if $G$ has an eulerian trail, i.e., $G$ has an open trail which contains all the vertices and edges of $G$ (and in view of the next result, this trail begins at one of the odd vertices and ends at the other).

*Theorem* (*Euler*). Let $G$ be a connected graph. Then $G$ is traversable if and only if $\in(G) = 2$.

Let $G$ be a hypo-traversable graph. Then $\in(G) \neq 2$, and $\in(G - v) = 2$ for each vertex $v$ of $G$. It is clear that $G$ is a block, and $\delta(G) \geq 2$. Also, $\in(G)$ is even and $0 \leq \in(G) \leq p$. From the first possible value we readily get the following.

*Theorem* 6. Let $G$ be any connected graph which has euler number 0. Then $G$ is hypo-traversable if and only if $G$ is a cycle.

*Proof.* The sufficiency is immediate, and for the necessity we note that $\in(G) = 0$ implies that $V(G) = \xi(G)$. Now $\in(G - v) = 2$ for any vertex $v$ of $G$ gives deg $v = 2$. By connectedness, $G$ has to be a cycle.

Now let $\in(G) = 2m$, $m \geq 2$, and let $G$ be hypo-traversable. Let $u \in \xi(G)$ and $v \in \theta(G)$. Then it can be seen that deg $u = 2m - 2$, $2m$ or $2m + 2$ and deg $v = 2m - 3$, $2m - 1$ or $2m + 1$, otherwise $\in(G - w) \neq 2$ for some vertex $w$ of $G$. This fact is useful in considering individual cases. Should $m = 2$, the possible values of deg $v$ will be 3 or 5 since $\delta(G) \geq 2$. It can be verified that for $p \leq 5$, cycles are the only hypo-traversable graphs. Figure 1 shows all graphs on 6 vertices which are hypo-traversable.

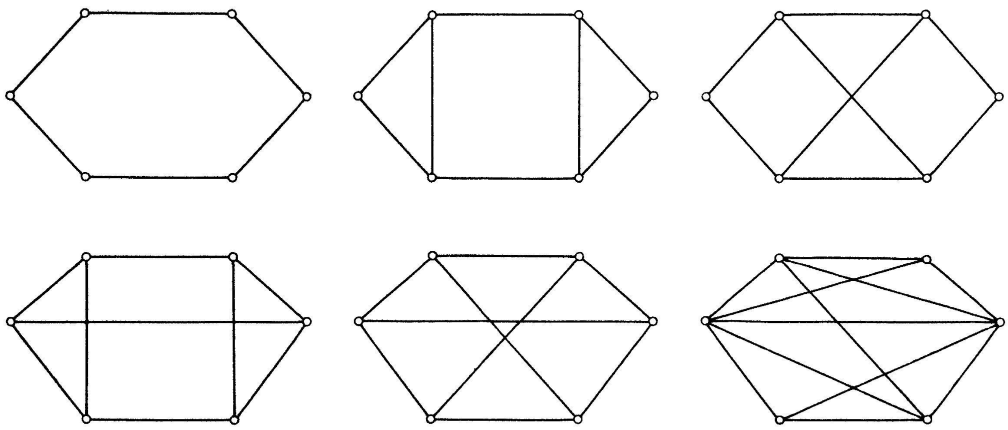

Figure 1
Hypo-traversable graphs on 6 vertices.

The preceding theorem dealt with the case when the graph had all vertices even. The next result treats graphs possessing no even vertices.

*Theorem* 7. Let $G$ be any connected graph having euler number $\in(G) = p(G) \geq 6$. Then $G$ is hypo-traversable if and only if $G$ is regular of degree $p - 3$.

*Proof.* Here $\xi(G) = \phi$ and $p = 2m = \in(G)$. By the above remarks, every vertex of $G$ is odd and has possible degrees $2m - 3$ or $2m - 1$. But if any vertex is adjacent with all the other $p - 1$ vertices, its deletion gives an eulerian graph. The necessity now follows.

Conversely, let $G$ be a connected $(p - 3)$-regular graph and $\in(G) = p(G) \geq 6$. Then $\in(G - v) = 2$ for all $v$, and the proof is complete.

*Theorem* 8. Let $G$ be a connected graph having euler number $\in(G) = p(G) - 1$, and let $p(G) \geq 5$. Then $G$ is hypo-traversable if and only if the even vertex $u$ of $G$ has degree $p - 3$, the vertices $a$ and $b$ that are nonadjacent with $u$ have degree $p - 4$, and every other vertex has degree $p - 2$.

*Proof.* Let $\xi(G) = \{u\}$, and assume that $G$ is hypo-traversable. Since every vertex adjacent with $u$ becomes even in the traversable graph $G - u$, we need deg $u = p - 3$. Let $a$ and $b$ be the vertices nonadjacent with $u$, and let $v \in \theta(G) - \{a, b\}$. Now the traversable graph $G - w$ contains exactly 2 odd vertices, for each $w \in V(G)$.

Hence $\deg v = p - 2$ and $\deg a = \deg b = p - 4$. For the sufficiency we note that $\in(G) \geq 4$, and by hypothesis, $\in(G - w) = 2$ for each vertex $w$ of $G$.

It is possible that a complete classification of hypo-traversable graphs may get involved with discussing individual cases, and this suggests scope for further research.

Let $G$ be a hypo-nontraversable graph, i.e., $\in(G) = 2$ and $\in(G - v) \neq 2$ for each vertex $v$. Moreover, since it is meaningful to require that $G - v$ be connected, we further assume that $G$ has no cutpoints and $p \geq 4$ (so that $\delta(G) \geq 2$). Designate the two odd vertices of $G$ as $a$ and $b$. If $a\,b$ is not an edge in $G$, then $\in(G - a)$ and $\in(G - b)$ are 4 or more. On the other hand, if $a$ and $b$ are adjacent, we must have $\deg a \geq 5$ and $\deg b \geq 5$. Now let $v \in \xi(G)$. This imposes the following restrictions: If $\deg v = 2$, then $v$ is adjacent with either both or neither of $a$ and $b$; if $\deg v = 4$, then $v$ is not simultaneously joined to both $a$ and $b$. These present a set of necessary conditions for $G$ to have the desired property, and it can be verified that they are also sufficient.

*Theorem* 9. Let $G$ be a block with $p \geq 4$. Then $G$ is hypo-nontraversable if and only if $\theta(G) = \{a, b\}$ and

(i) $a\,b \in E(G) \Rightarrow \deg a \geq 5$ and $\deg b \geq 5$,

(ii) $\deg v = 2 \Rightarrow v$ is joined to both or neither of $a$, $b$, and

(iii) $\deg v = 4 \Rightarrow v$ is not joined to both $a$ and $b$.

In [1] a traversable graph $G$ is called *randomly traversable from a vertex* $v$ if every trail in $G$ with initial vertex $v$ can be extended to an eulerian trail of $G$. Clearly, a traversable graph can be randomly traversable from $k = 0$, 1 or 2 vertices, and we may, as before, denote this class of graphs as $RT(k)$, where $RT(2)$ will refer to the class of *randomly traversable graphs*. It was also proved in [1] that if $a$ and $b$ are the two odd vertices of a traversable graph $G$, then $G$ is randomly traversable from $a$ if and only if every cycle of $G$ contains $b$. Moreover, a graph $G$ is in $RT(2)$ if and only if the two odd vertices of $G$ lie on every cycle of $G$. This suggests the problem of studying *hypo-$RT(k)$* and *hypo-non$RT(k)$* graphs.

We conclude by presenting a complete classification of $RT(2)$ graphs.

*Theorem* 10. Let $G$ be a traversable graph with $\theta(G) = \{a, b\}$. Then $G$ is randomly traversable if and only if $G$ is homeomorphic from $K_2$, $K(2, 2m - 1)$ or $K(2, 2m) + a\,b$, where $m \geq 1$.

*Proof.* It is obvious that the graphs described are randomly traversable. To prove the converse, first we note that if $\deg a = 1$, then any $b - a$ path must be $G$ itself, otherwise there exists a cycle which avoids $a$ or $b$. Thus, $\deg b = 1$, and the graph $G$ is homeomorphic from $K_2$. So we assume that each of $a$ and $b$ has degree at least 3.

Let $v$ be any vertex of $G$ other than $a$ or $b$. Since $G$ is connected, there exist $v - a$ and $v - b$ paths. Clearly these paths have $v$ as their only common vertex otherwise some cycle of $G$ avoids $a$ or $b$. Moreover, the union of these paths gives an $a - b$ path which contains $v$. With every vertex $v \in V(G) - \theta(G)$ we can associate an $a - b$ path $P(v)$ such that $P(v)$ contains $v$. Let us consider the collection of all $a - b$ paths, where, for obvious reasons, any two paths are disjoint, i.e., the only vertices common to them are $a$ and $b$. So $P(v)$ is unique, and the union of all these paths must be $G$ itself. We therefore conclude that every vertex other than $a$ and $b$ has degree 2, and $\deg a = \deg b$ is odd. Also, if $a$ and $b$ are adjacent, then $G - ab$ is homeomorphic from $K(2, 2m)$; and if $a, b$ are nonadjacent, then $G$ is homeomorphic from $K(2, 2m - 1)$, where $m \geq 1$.

S.F. Kapoor[1]), Western Michigan University, USA

REFERENCES

[1] G. CHARTRAND and A. T. WHITE, *Randomly Traversable Graphs*, El. Math. *25*, 101–107 (1970).
[2] F. HARARY, *Graph Theory* (Addison-Wesley, Reading 1969).
[3] J. MITCHEM, *Hypo-Properties in Graphs, The Many Facets of Graph Theory* (G. Chartrand and S. F. Kapoor editors), Lecture Notes in Mathematics No. 110 (Springer-Verlag, Berlin 1969), p. 223–230.
[4] O. ORE, *A Problem Regarding the Tracing of Graphs*, El. Math. *6*, 49–53 (1951).

1) Research partially supported by National Science Foundation grant GP 9435.

# Kleine Mitteilungen

## New Quadratic Forms with High Density of Primes

Let $p_{min}$ be the smallest prime contained in a quadratic form of the shape $f(x) = A x^2 + A x - C$ and let $n_{icp}$ be the number of initial consecutive primes of $f(x)$, then, by means of a CDC 6400 computer, all $f(x) = A x^2 + A x - C$ were investigated for $A < 10$, $C < 2.10^5$, and $p_{min} > 47$. In Table 1, the number below $C$ is the number of all primes of $f(x)$ for $x < 100$, and $p_{min}$ is the number in parentheses.

For each form $x^2 + x - C$ we have also a form $9 y^2 + 9 y - (C - 2)$, because the substitution $x = 3y + 1$ transforms $x^2 + x - C$ into $9 y^2 + 9 y - (C - 2)$; hence, each third term of $x^2 + x - C$ (starting with the second) belongs to $9 y^2 + 9 y - (C - 2)$. Similarly, for each form $2 x^2 - C$ we have also a form $8 z^2 + 8 z - (C - 2)$, because the substitution $x = 2z + 1$ transforms $2 x^2 - C$ into $8 z^2 + 8 z - (C - 2)$; hence, each second term of $2x^2 - C$ (starting with the second) belongs to $8 z^2 + 8 z - (C - 2)$. For the forms $2 x^2 - 119131$ and $2 x^2 - 186871$, related to the forms with $A = 8$ in Table 1, we have 64 and 61 primes, respectively, for $x < 100$.

Table 1 gives the impression that there might be no forms with $A = 4$. This is not so. In a test run with $A < 10$, $10^8 - 5000 < C < 10^8$, and $p_{min} > 47$, the forms $x^2 + x - 99995659$, $9 x^2 + 9 x - 99995657$, and $4 x^2 + 4 x - 99996937$ were discovered, all with $p_{min} = 53$.

The form $x^2 + x - 53509$ with $p_{min} = 61$ is due to N.G.W.H. Beeger [1] in 1938, the forms $x^2 + x - 90073$ with $p_{min} = 53$ and $x^2 + x - 169933$ with $p_{min} = 59$ are due to the author [2] in 1967.

Two hundred years ago, Euler published his famous quadratic form $x^2 + x + 41$ with $p_{min} = 41$ and $n_{icp} = 40$. This form was believed to have the highest density of primes of all quadratic forms $A x^2 + B x \pm C$ discovered till now. Many forms were found with $p_{min} > 41$ and the second differences greater than 2; but the corresponding