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A Criterion for n-Fold Transitivity of Transformation Groups

Let G be a group and let X be a nonempty set. An *action* $*$ on X is a function $*$: $G \times X \to X$ such that for every $g, h \in G$ and $x \in X$, (*i*) $(gh) * x = g * (h * x)$ and (*ii*) $1 * x = x$.

A triple $(G, X, *)$ where $*$ is an action of G on X is called a *transformation group*. For $S \subseteqq X$ the stability subgroup of S is $G_S = \{g \in G \mid g * s = s \text{ for every } s \in S\}$. (We will write G_x instead of $G_{\{x\}}$.)

If n is a positive integer, we say that G is *n-fold transitive* whenever for every two sequences $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ each consisting of n distinct elements of X, there exists $g \in G$ such that $g * x_i = y_i$ for every $_i = 1, 2, \ldots, n$.

We note that if $*$ is an action of G on X, then for any $S \subseteqq X$, $*$ induces an action of G_S on $X - S$.

The next theorem is well known (see, for example, [1], Theorem 9.1).

Theorem 1: Let $(G, X, *)$ be transitive. Then for $n \geq 2$, $(G, X, *)$ is n-fold transitive iff there exists an $x \in X$ such that $(G_x, X - \{x\}, *)$ is $(n-1)$-fold transitive.

It is our purpose in this note to derive a corollary (Theorem 2) of this theorem which is sometimes more convenient to use. The essential idea is to replace the transitive condition on $(G, X, *)$ by a restriction on the stability subgroups.

Lemma 1: If $(G, X, *)$ is a transformation group, then $(G, X, *)$ is 2-fold transitive iff there exists an $x \in X$ such that $G_x \neq G$ and $(G_x, X - \{x\}, *)$ is transitive.

Proof: Clearly if $(G, X, *)$ is 2-fold transitive then the given condition holds for any $x \in X$.

Now suppose $x \in X$ such that $G_x \neq G$ and $(G_x, X - \{x\}, *)$ is transitive. Let $y, z \in X$. If $y, z \in X - \{x\}$, then there exists $g \in G_x$ such that $g * y = z$. If $y = z = x$, then $1 * y = z$. If $y = x$ and $z \neq x$, then since $G_x \neq G$, there exists $h \in G$ such that $h * x \neq x$. So there is an $r \in G_x$ such that $r * (h * x) = z$ and so $(rh) * x = z$. If $y \neq x$, $z = x$ and h is as before, then there exists $t \in G_x$ such that $t * y = h * x = h * z$ so that $(h^{-1} t) * y = z$. Hence $(G, X, *)$ is transitive so that by Theorem 1 it is 2-fold transitive.

Lemma 2: Let $n \geq 2$ and $|X| > 1$. Then $(G, X, *)$ is n-fold transitive iff there exists an $x \in X$ such that $G_x \neq G$ and $(G_x, X - \{x\}, *)$ is $(n-1)$-fold transitive.

Proof: Assume $G_x \neq G$ and $(G_x, X - \{x\}, *)$ is $(n-1)$-fold transitive. Then by Lemma 1, $(G, X, *)$ is transitive and hence by Theorem 1 it is n-fold transitive. If $(G, X, *)$ is n-fold transitive, then the given condition holds for all $x \in X$.

Theorem 2: For $|X| \geq n \geq 2$, $(G, X, *)$ is n-fold transitive iff there exists $S \subseteqq X$ with $S = \{t_1, t_2, \ldots, t_{n-1}\}$ such that if $S_k = \{t_1, t_2, \ldots, t_k\}$ for each $k = 1, 2, \ldots, n-1$, then

a) $G_{t_1} \neq G$ and $G_{S_k} \neq G_{S_{k+1}}$ for all $k = 1, 2, \ldots, n-1$; and

b) $(G_S, X - S, *)$ is transitive.

Proof: Since any n-fold transformation group clearly satisfies (a) and (b), we need only show the other half.

The case $n = 2$ is the content of Lemma 1.

Suppose the theorem holds for all integers greater than one and less than n. Let $S \subseteqq X$ be $S = \{t_1, t_2, \ldots, t_{n-1}\}$ such that Conditions (a) and (b) hold. Then $S^* = \{t_2, \ldots, t_{n-1}\}$ satisfies the conditions of the theorem for the transformation group $(G_{t_1}, X - \{t_1\}, *)$ and hence this transformation group is $(n-1)$-fold transitive. But then by Lemma 2, $(G, X, *)$ is n-fold transitive.

We next consider an application of this result. Let k be a field and let G be the group $GL(k, 2)$ of all nonsingular 2×2 matrices over k. Let $*$ be the action of G on $k \cup \{\infty\}$ defined by

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} * z = \begin{cases} \dfrac{\alpha z + \beta}{\gamma z + \delta}, & \text{if } z \neq \infty,\ \gamma z + \delta \neq 0 \\ \infty, & \text{if } z \neq \infty,\ \gamma z + \delta = 0 \\ \alpha/\gamma, & \text{if } z = \infty,\ \gamma \neq 0 \\ \infty, & \text{if } z = \infty,\ \gamma = 0. \end{cases}$$

We will apply the previous result to show that $(G, X, *)$ is 3-fold transitive. First we note the following special case of Theorem 2 obtained by letting $n = 3$.

Theorem 3: For $|X| \geq 3$, $(G, X, *)$ is 3-fold transitive iff there exist $x, y \in X$ such that $G_x \neq G$, $G_{\{x,y\}} \neq G_x$ and $(G_{\{x,y\}}, X - \{x, y\}, *)$ is transitive.

Note that $G_{\{x,y\}} = G_x \cap G_y$. It is easy to see that

$$G_\infty = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \;\middle|\; a, b, c \in k,\ ac \neq 0 \right\}$$

and

$$G_0 = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in k,\ ac \neq 0 \right\}.$$

So

$$G_{\{0,\infty\}} = G_0 \cap G_\infty = \left\{ \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \;\middle|\; a, c \in k,\ ac \neq 0 \right\}.$$

Hence $G_0 \neq G$, $G_{\{0,\infty\}} \neq G_0$. It is also clear that $(G_{\{0,\infty\}}, k-\{0\}, *)$ is transitive, for if $x \neq 0$ and $y \neq 0$, then

$$\begin{bmatrix} y & 0 \\ 0 & x \end{bmatrix} * x = y.$$

Hence by Theorem 3, $(G, X, *)$ is 3-fold transitive. We note that $(G, X, *)$ is not 4-fold transitive, for then $(G_{\{0,\infty\}}, k-\{0\}, *)$ would be 2-fold transitive.

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On 1-Factorability and Edge-Colorability of Cartesian Products of Graphs

There is no characterization of 1-factorable graphs. Thus, it is natural that many of the results on this topic have been the determination of classes of 1-factorable graphs. The object of this paper is to present a sufficient condition for the 1-factorability of the cartesian product of two graphs. We begin with some notation and definitions.

The vertex set of a graph G will be denoted by $V(G)$ and its edge set by $E(G)$. In this paper we consider only finite, undirected graphs without loops or multiple edges. Let G and H be two nonempty graphs for which $V(G) = V(H)$ and $E(G) \cap E(H) = \Phi$; then the graph G' is the *sum* of G and H, written $G' = G + H$, if $V(G') = V(G)$ and $E(G') = E(G) \cup E(H)$. A 1-*factor* of a graph G is a spanning 1-regular subgraph of G. A graph is 1-*factorable* if it can be expressed as a sum of edge-disjoint 1-factors. The *cartesian product* (or *product*) of the graph G with the graph H, denoted by $G \times H$, is defined by: $V(G \times H) = V(G) \times V(H)$; $E(G \times H) = \{[(u_1, v_1), (u_2, v_2)] \mid u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)\}$.

An assignment of n colors to the edges of a nonempty graph G so that adjacent edges are colored differently is an *n-edge-coloring* of G. The minimum n for which a graph G is n-edge-colorable is its *edge-chromatic number* $\chi_1(G)$. By a theorem of Vizing [2], the edge-chromatic number $\chi_1(G)$ of a graph G is bounded by: $\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. If G is regular, then G is 1-factorable if and only if $\chi_1(G) = \Delta(G)$. Hence any theorem concerning the 1-factorability of regular graphs has as an immediate corollary a result concerning edge-colorability, which is useful since there is also no characterization of those graphs which are $\Delta(G)$-edge-colorable. For other notations and definitions, we follow [1].

If K_2 denotes the complete graph on two vertices, then $K_2 \times H$, where H is any regular graph, is shown to be 1-factorable in the following lemma.