Zeitschrift: Elemente der Mathematik
Band: 29 (1974)
Heft: 6

Artikel: A triangle transformation
Autor: Bottema, O.
DOI: https://doi.org/10.5169/seals-29907

stehende Polytop ist wiederum ein reguläres 16-Zell mit der Kantenlänge $\sqrt{2}$, also den beiden anderen kongruent.

Beide Beispiele stehen nach H. Groemer [3] in engem Zusammenhang damit, dass es Parkettierungen des E_4 mit lauter regulären 24-Zellen (alle in gleicher Drehlage) bzw. mit lauter regulären 16-Zellen (in drei verschiedenen Drehlagen) gibt (vgl. H. S. M. Coxeter [1], S. 296). Da solche Parkettierungen mit regulären 120-Zellen nicht existieren, ist es vermutlich schwierig, die Zerlegungsgleichheit des 120-Zells mit einem Hyperwürfel explizit zu realisieren.

P. Mürner, Gymnasium Interlaken

LITERATURVERZEICHNIS

[1] H. S. M. COXETER, *Regular Polytopes*, (Methuen, London 1948).
[2] M. DEHN, *Über den Rauminhalt*, Math. Ann. *55*, 465–478 (1901).
[3] H. GROEMER, *Über Würfel- und Raumzerlegungen*, El. Math. *19*, 25–27 (1964).
[4] H. HADWIGER, *Translationsinvariante, additive und schwachstetige Polyederfunktionale*, Arch. Math. *3*, 387–394 (1952).
[5] H. HADWIGER, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer-Verlag 1957.
[6] B. JESSEN, *Zur Algebra der Polytope*, Nachr. Akad. Wiss. Göttingen II 1972, 47–53.
[7] B. JESSEN, und A. THORUP, *The Algebra of Polytopes in Affine Spaces* (1973, noch unpubliziert).
[8] H. LEBESGUE, *Sur les subdivisions des polyèdres réguliers en polyèdres réguliers*, Publ. math. Univ. Belgrade *6-7*, 183–188 (1938).
[9] H. LEBESGUE, *Sur l'équivalence des polyèdres, en particulier des polèdres réguliers, et sur la dissection des polyèdres réguliers en polyèdres réguliers*, Ann. Soc. Polon. Math. *17*, 193–226 (1939).

A Triangle Transformation

1. The configuration of a triangle on the sides of which polygons of a certain kind are described is a much studied theme in elementary geometry. The following variant does not seem to be well-known.

On the sides of a given triangle ABC similar isosceles triangles BCA, CAB, ABC, are constructed (Fig. 1), with bases BC, CA, AB, all outward or all inward, the base angle φ being taken positive or negative respectively $(-\pi/2 < \varphi < \pi/2)$. The operation thus defined which transforms the triangle $\Delta = ABC$ into $\Delta_1 = A_1B_1C_1$ will be denoted by $T(\varphi)$.

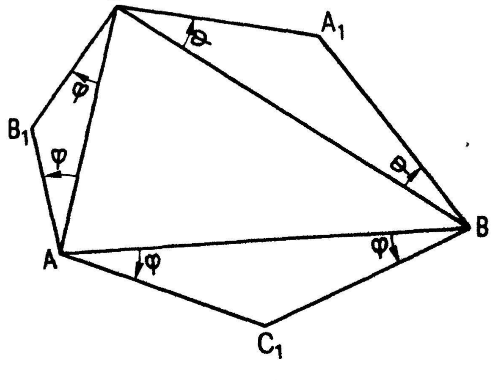

Fig. 1

The sides of Δ, Δ_1 are a, b, c and a_1, b_1, c_1, the oriented areas F and F_1, $S = a^2 + b^2 + c^2$, $S_1 = a_1^2 + b_1^2 + c_1^2$. As $A_1 B = A_1 C = (1/2)\, a \cos^{-1} \varphi$ etc., the cosine rule gives

$$
\begin{aligned}
4 a_1^2 &= \cos^{-2} \varphi \{b^2 + c^2 - 2 b c \cos (\alpha + 2 \varphi)\} \\
&= \cos^{-2} \varphi \{b^2 + c^2 - (b^2 + c^2 - a^2)\ (1 - 2 \sin^2 \varphi) + 2 b c \sin \alpha \sin 2\varphi\} \\
&= (2 b^2 + 2 c^2 - a^2)\, t^2 + a^2 + 8 F t\,, \qquad (1.1)
\end{aligned}
$$

with $t = \tan\varphi$. For $4 b_1^2$ and $4 c_1^2$ we obtain analogous formulas. Hence

$$S_1 = \frac{1}{4}\, (3t^2 + 1)\ S + 6\, t\, F\,. \qquad (1.2)$$

For the area of Δ_1 we have

$$F_1 = F + \sum (\text{area } BA_1C) - \sum (\text{area } B_1AC_1)\,. \qquad (1.3)$$

Obviously

$$\text{area } BA_1C = \frac{1}{4}\, t\, a^2\,, \qquad (1.4)$$

and

$$
\begin{aligned}
\text{area } B_1AC_1 &= \frac{1}{8} \cos^{-2}\varphi \cdot bc\ \sin (\alpha + 2\varphi) \\
&= \frac{1}{8} \cos^{-2}\varphi \{2 F (\cos^2\varphi - \sin^2\varphi) + (b^2 + c^2 - a^2)\ \sin\varphi \cos\varphi\} \\
&= \frac{1}{4}\ (1 - t^2)\ F + \frac{1}{8}\ t\, (b^2 + c^2 - a^2)\,. \qquad (1.5)
\end{aligned}
$$

Hence from 1.4 and 1.5

$$\text{area } BA_1C - \text{area } B_1AC_1 = \frac{1}{4}\, (t^2 - 1)\ F + \frac{1}{8}\, t\, (3 a^2 - b^2 - c^2)\,,$$

and 1.3 gives us

$$F_1 = \frac{1}{4}\, (3\, t^2 + 1)\ F + \frac{1}{8}\, t\, S \qquad (1.6)$$

The formulas (1.2) and (1.6) express F_1 and S_1 as linear functions of F and S. They have been derived recently in a different context, for $\varphi > 0$ only and with a less simple proof of (1.6) by Toscano [1]. They are valid for all values of φ; if F_1 and F have different signs the orientation of Δ_1 is the opposite of that of Δ.

2. F and S are related to the well-known Brocard's angle ω of the triangle [2]. We have, accepting a negative value of ω for a triangle with negative orientation

$$\cot \omega = S/4 F \qquad (2.1)$$

One always has $\cot^2 \omega \geqq 3$, or $-\pi/6 \leqq \omega \leqq \pi/6$, with equality only for equilateral triangles.

From (1.2) and (1.6) it follows

$$\cot \omega_1 = \frac{(3\, t^2 + 1) \cot \omega + 6\, t}{2\, t \cot \omega + (3\, t^2 + 1)}\,, \qquad (2.2)$$

which means that ω_1 depends only on t and ω. *If two triangles are equibrocardian their transforms by $T(\varphi)$ are equibrocardian as well.*

The right-hand side of (2.2) is a linear function of cot ω the determinant of which is $(3t^2 - 1)^2$. Hence it is singular only if $t = \pm \sqrt{1/3}$; then for any ω we have $\cot \omega_1 = \pm \sqrt{3}$. Therefore *by $T(\pm \pi/6)$ any triangle is tranformed into an equilateral triangle,* a well-known theorem.

We divide the set of all triangles into classes $K(\omega)$, the elements of a class being the triangles with a given ω. The classes $K(\pm \pi/6)$ contain the equilateral triangles, $K(0)$ is the class of degenerated triangles. The conclusion is: any $T(\varphi)$ permutes the classes $K(\omega)$.

(2.2) may be written as follows

$$(3\,t^2 + 1)\;(\cot \omega_1 - \cot\omega) + 2\,t\;\cot\omega_1\;\cot\omega - 6\,t = 0 \tag{2.3}$$

$\omega_1 = \omega$ implies

$$t\,(\cot^2\omega - 3) = 0 \tag{2.4}$$

Hence the two classes of equilateral triangles are invariant for any $T(\varphi)$ and any class $K(\omega)$ is invariant for $T(0)$. Both properties are obvious: $T(0)$ transforms a triangle into that of the midpoints of its sides.

If ω and ω_1 are given, (2.3) is a quadratic equation for t:

$$3\,t^2\,(\cot\omega_1 - \cot\omega) + 2\,t\,(\cot\omega_1 \cot\omega - 3) + (\cot\omega_1 - \cot\omega) = 0 \tag{2.5}$$

Its discriminant d satisfies

$$d = (\cot^2\omega_1 - 3)\;(\cot^2\omega - 3) \gtreqless 0 \tag{2.6}$$

Hence 2.5 has real roots t_1, t_2; *in general, two transformations $T(\varphi_1)$, $T(\varphi_2)$ exist which transform a given class $K(\omega)$ into a given class $K(\omega_1)$.* One always has $t_1 t_2 = 1/3$.

If t_0 is a root of (2.5) then $-t_0$ is one of the equation with ω and ω_1 interchanged. *If $T(t)$ transforms $K(\omega)$ into $K(\omega_1)$ the two transforming $K(\omega_1)$ into $K(\omega)$ are $T(-t)$ and $T(-1/3\,t)$.* In particular: Δ and $T(-t)\,\{T(t)\cdot\Delta\}$ are equibrocardian.

If $T(t_1)\cdot\Delta = \Delta_1$ and $T(t_2)\cdot\Delta_1 = \Delta_2$, we have

$$\cot\omega_2 = \frac{\{3\,(t_1 + t_2)^2 + (3\,t_1 t_2 + 1)^2\}\cos\,\omega + 6\,(t_1 + t_2)\;(3\,t_1 t_2 + 1)}{2\,(t_1 + t_2)\;(3\,t_1 t_2 + 1)\,\cos\,\omega + \{3(t_1 + t_2)^2 + (3\,t_1 t_2 + 1)^2\}}$$

or, if $3\,t_1 t_2 + 1 \neq 0$,

$$\cos\omega_2 = \frac{(3\,t_{12}^2 + 1)\,\cos\,\omega + 6\,t_{12}}{2\,t_{12}\cos\,\omega + (3\,t_{12}^2 + 1)}, \tag{2.7}$$

with

$$t_{12} = \frac{t_1 + t_2}{3\,t_1 t_2 + 1}, \tag{2.8}$$

giving *the multiplication rule* for two transformations of the set: $T(t_2)\cdot\{T(t_1)\cdot\Delta\}$ and $T(t_{12})\cdot\Delta$ are equibrocardian triangles. The multiplication is *commutative.* If Δ is a given triangle with Brocard angle ω there are two transformations T such that $\omega_1 = 0$ and hence $\cot\omega_1 = \infty$. From (2.5) it follows that they are $T(t_i)$, $i = 1,2$, such that t_i are the roots of

$$3\,t^2 + 2\,t\cot\omega + 1 = 0\,, \tag{2.9}$$

that is

$$t_i = \frac{1}{3}\{-\cot\omega \pm (\cot^2\omega - 3)^{1/2}\}. \tag{2.10}$$

If on the sides of a triangle we construct isosceles triangles with tan $\varphi = t_1$ or t_2 the points A_1, B_1, C_1 are collinear. As $t_1 t_2 = 1/3$, $t_1 + t_2 = (-2/3) \cot \omega$ we have $\varphi_1 + \varphi_2 = \omega - \pi/2$.

3. Properties of the transformation $T(t)$ may also be found by vector algebra. We denote the point P of the plane by the vector $\overline{P}$ from the origin O to P; the unit vector perpendicular to the plane by $\bar{e}$.

We obtain

$$2\overline{A}_1 = (\overline{B} + \overline{C}) + t\bar{e}\times(\overline{B} - \overline{C}),\ 2\overline{B}_1 = (\overline{C} + \overline{A}) + t\,\bar{e}\times(\overline{C} - \overline{A}),$$
$$2\overline{C}_1 = (\overline{A} + \overline{B}) + t\bar{e}\times(\overline{A} - \overline{B}) \tag{3.1}$$

As $\overline{A}_1 + \overline{B}_1 + \overline{C}_1 = \overline{A} + \overline{B} + \overline{C}$, we conclude: *the triangles Δ and $T\cdot\Delta$ have the same centroid.*

Furthermore if $\Delta_1 = T(t_1)\cdot\Delta$ amd $\Delta_2 = T(t_2)\cdot\Delta_1$ we have

$$2\overline{A}_2 = (\overline{B}_1 + \overline{C}_1) + t_2\bar{e}\times(\overline{B}_1 - \overline{C}_1)$$

or

$$4\overline{A}_2 = (2\overline{A} + \overline{B} + \overline{C}) + (t_1 + t_2)\,\bar{e}\times(\overline{C} - \overline{B}) + t_1 t_2\,(2\overline{A} - \overline{B} - \overline{C}), \tag{3.2}$$

and analogously for $4\,B_2$ and $4\,C_2$. If we take the origin O at the centroid G of ABC we obtain

$$4\overline{A}_2 = (1 + 3\,t_1 t_2)\,\overline{A} + (t_1 + t_2)\,\bar{e}\times(\overline{C} - \overline{B}). \tag{3.3}$$

Two special cases are of interest. If $t_1 + t_2 = 0$, $-t_2 = +t_1 = t$ one has

$$4\overline{A}_2 = (1 - 3\,t^2)\,\overline{A},\ 4\overline{B}_2 = (1 - 3\,t^2)\,\overline{B},\ 4\overline{C}_2 = (1 - 3\,t^2)\,\overline{C}. \tag{3.4}$$

Therefore: *If on the sides of ABC we describe outward (inward) isosceles triangles with base angle φ and then on those of $A_1B_1C_1$ inward (outward) isosceles triangles with the same angle φ then $A_2B_2C_2$ and ABC are homothetic with respect to the centroid G, with the scale factor* (1/4) (1 − 3 tan² φ). The two triangles are congruent if $\tan\varphi = \pm\sqrt{5/3}$.

If $1 + 3\,t_1 t_2 = 0$, $t_1 + t_2 = t$ we obtain

$$4\overline{A}_2 = t\bar{e}\times(\overline{C} - \overline{B}),\quad 4\overline{B}_2 = t\bar{e}\times(\overline{A} - \overline{C}),\quad 4\overline{C}_2 = t\bar{e}\times(\overline{B} - \overline{A}). \tag{3.5}$$

Hence the medians of $A_2B_2C_2$ are perpendicular to the sides of ABC and equal to (3/8) ta, (3/8) tb, (3/8) tc; from this is follows that the sides of $A_2B_2C_2$ are proportional to the medians of ABC with ratio $t/2$.

O. Bottema, Delft

REFERENCES

[1] L. TOSCANO, *Sur deux suites récurrentes du second ordre, linéaires et homogènes*, Математички Весник, *9(24)*, СТР. 131–140 (1972).

[2] Cf. T. LALESCO, *La géométrie du triangle*, (Paris, 1937), 48–55, 117–120; R. A. JOHNSON, *Advanced Euclidean Geometry* (Dover Publ. New York, 1960), Ch. XVI, The Brocard configuration, 263–286, Ch. XVII, Equibrocardal triangles, 287–301.