Inner illumination of convey polytopes

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Inner Illumination of Convex Polytopes

1. *Introduction.* An n-polytope P is said to be illuminated by its vertices, if for every vertex x of P there is another vertex y of P such that the line segment joining x and y meets the interior of P. Hadwiger in [1], introduced the notion of polytopes illuminated by their vertices and asked whether such polytopes must have at least $2n$ vertices. Recently, Mani [2], proved that for $n \leq 7$ the answer to Hadwiger's problem is affirmative, while for higher dimensions he showed that there are n-polytopes P, that are illuminated by their vertices having about $n + 2\sqrt{n}$ vertices. Mani obtained the exact lower bound $k(n)$ for the number of vertices in an n-polytope P which is illuminated by its vertices. Mani's proof is based on the notion of a set of vertices lying opposite a given vertex of P. The proof proceeds by showing that if for some vertex x of the n-polytope P there is more than one vertex lying opposite x then $f^0(P) \geq k(n)$, while if for every vertex x of P there is at most one vertex lying opposite x then $f^0(P) \geq 2n$. For the second part of the proof, results and tools from algebraic topology as well as some combinatorial lemmas (Propositions 4 and 5) were used. In this note, we present an alternative proof to this part that avoids using the lemmas and the algebraic topology.

The notation used in this note will be the same as Mani's; we will only repeat those definitions and notation that are used in our proof.

We denote by $\Delta^0 P$ the set of vertices of the polytope P and by $f^0(P)$ their number.

A set $V \subset \Delta^0 P$ illuminates itself if for every v in V there is another vertex v' in V that illuminates v in P.

A set $Y \subset \Delta^0 P$ lies opposite the vertex x, if x illuminates every vertex y in Y and $\Delta^0 P \sim (\{x\} \cup Y)$ illuminates itself.

We set $\gamma(x, P) = \max\{\text{card } Y : Y \text{ lies opposite } x \text{ in } P\}$.

2. *Proof of the Theorem.*

Theorem: If $P \subset E^n$ is illuminated by its vertices, then either $\gamma(x, P) \geq 2$ for some vertex x of P or $f^0(P) \geq 2n$.

Proof. Assume first that for some vertex x of P, $\gamma(x, P) = 0$. Since P is illuminated by its vertices there is a vertex x' of P that illuminates x. Since $\gamma(x, P) = 0$, the set $C = \Delta^0 P \sim \{x, x'\}$ does not illuminate itself. Let A be the set of all vertices in C that are not illuminated in C. Let Y be the set of all vertices y in A that are illuminated in P by x, and let $Y' = A \sim Y$. Obviously, if y' is in Y', then x' illuminates y'. If Y' is empty, then the set $\{x'\} \cup \{Y\} \subset \Delta^0 P$ lies opposite x, in contradiction to the assumption that $\gamma(x, P) = 0$. If $Y \neq \emptyset$, then the set Y lies opposite x and again we obtain a contradiction. Hence Y must be empty. Since $A \neq \emptyset$, and since the set $\{x\} \cup A$ lies opposite x', we have $\gamma(x', P) \geq 2$.

We may therefore assume that $\gamma(x, P) = 1$ for every vertex x in P. Let G be a graph with vertex set $V(G) = \Delta^0 P$, and (x, y) in $E(G)$ if $\{y\}$ lies opposite x. (Obviously, if $\{y\}$ lies opposite x then $\{x\}$ lies opposite y.) We will show that G has a 1-factor. This will be done in two steps.

(1) We show first that the valence of every vertex x of G is at most two. Indeed if $\{x_1, \ldots, x_k\}$, $k > 2$, is the set of all vertices of G which are connected by an edge to x_0, the set $D = \Delta^0 P \sim \{x_0, x_1, \ldots, x_k\}$ cannot illuminate itself. Let $A \subseteq D$ be the set of vertices in D that are not illuminated in D and not illuminated by x_0. If $A = \emptyset$ then we would have $\gamma(x_0, P) \geq k$ (the set $\{x_1, \ldots, x_k\}$ and the vertices of D illuminated by x_0 would lie opposite x_0), hence we may assume that $A \neq \emptyset$. Let d be in A. Since x_1 lies opposite x_0, d must be illuminated by some x_j, $1 < j \leq k$. Since x_j also lies opposite x_0, d must be illuminated by some x_m with $m \neq j$. Without loss of generality, we may assume that d is illuminated by $\{x_1, x_2\}$. If $A = \{d\}$, then it is easily seen that $\{x_1, x_2\}$ lies opposite d and we would have $\gamma(d, P) \geq 2$, in contradiction to the assumption that $\gamma(x, P) = 1$ for every vertex in P. Hence $A = \{d, e, \ldots, z\}$. If every vertex in A other than d is illuminated by some x_j with $j > 2$, then again $\{x_1, x_2\}$ would lie opposite d. We conclude that A must contain a vertex c that illuminates $\{x_1, x_2\}$. It is a simple matter to check that $\{d, c\}$ lies opposite x_1 and we would have $\gamma(x_1, P) \geq 2$. This establishes our claim.

(2) Let x_0 have valence 2 in G. Let x_1, x_2 be the two vertices such that x_i lies opposite x_0, $i = 1,2$. The set $D = \Delta^0 P \sim \{x_0, x_1, x_2\}$ does not illuminate itself. Let $A \subseteq D$ be the set of vertices in D that are not illuminated in D and not illuminated by x_0. If $A = \emptyset$, we would have $\gamma(x_0 P) \geq 2$, hence $A \neq \emptyset$. Since x_2 lies opposite x_0, and since for every d in A, d is not illuminated in D and not illuminated by x_0, d must be illuminated by x_1. Hence A lies opposite x_1 and by a similar argument, A lies opposite x_2. If card $A > 1$, then we would have $\gamma(x_1, P) \geq 2$. Therefore $A = \{d\}$ and d is connected by an edge to x_1 and x_2.

From (1) and (2) it follows that G is the disjoint union of edges and 4-cycles. Therefore G has a 1-factor. Let F be a facet of P. F contains at least n vertices of P. Since no two vertices of F illuminate each other, $\Delta^0 F$ is an independent set of vertices in G. Since G has a 1-factor, G must have at least $2\,n$ vertices.

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