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On Inscribed Circumscribed Conics

There is a well known geometric theorem due to Euler which states the following:

Theorem 1 (Euler). *Given a triangle inscribed in a circle of radius R and circumscribed about a circle of radius r , then*

$$R^2 - d^2 = 2 r R, \quad (1)$$

where d is the distance between the circumcenter and the incenter of the triangle.

In this note first we generalize Euler's Theorem as follows:

Theorem 2. *Let \mathfrak{C} be a circle about O with radius R and let \mathfrak{E} be an ellipse contained in \mathfrak{C} with semi-minor axis b and foci F_1, F_2 . Set $d_1 = \overline{OF_1}$, $d_2 = \overline{OF_2}$. Then there exists a triangle inscribed in \mathfrak{C} and circumscribed about \mathfrak{E} , if and only if*

$$(R^2 - d_1^2)(R^2 - d_2^2) = 4 b^2 R^2. \quad (2)$$

Proof. Let \mathfrak{R} and \mathfrak{Q} be two conics in the projective plane, given in homogeneous coordinates by

$$X^t A X = 0 \quad \text{and} \quad X^t B X = 0 \quad (3)$$

respectively, so that \mathfrak{R} contains \mathfrak{Q} . It is known (cf. [1] p. 279) that a necessary and sufficient condition for the existence of a triangle inscribed in \mathfrak{R} and circumscribed about \mathfrak{Q} is

$$\theta^2 = 4 \Delta \theta', \quad (4)$$

where Δ , θ , θ' , (and Δ'), are determined by

$$\det(A + \lambda B) \equiv \Delta + \theta \lambda + \theta' \lambda^2 + \Delta' \lambda^3. \quad (5)$$

It can be shown that

$$\Delta = \det A, \quad \theta = \text{tr}[(\text{adj } A)B], \quad \theta' = \text{tr}[(\text{adj } B)A], \quad (\Delta' = \det B), \quad (6)$$

where tr denotes the trace and adj the matrix of cofactors. Thus, the condition in (4) takes the form

$$\text{tr}^2[(\text{adj } A)B] = 4 \det A \cdot \text{tr}[(\text{adj } B)A]. \quad (7)$$

In our case let the ellipse and the circle be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (8)$$

and

$$(x - p)^2 + (y - q)^2 = R^2, \quad (9)$$

respectively. Then the non-singular symmetric matrices in (7) are

$$A = \begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -p \\ 0 & 1 & -q \\ -p & -q & p^2 + q^2 - R^2 \end{pmatrix}. \quad (10)$$

Therefore, in this particular case, (7) reads

$$(R^2 + a^2 + b^2 - p^2 - q^2)^2 = 4(a^2 b^2 - a^2 q^2 - b^2 p^2 + a^2 R^2 + b^2 R^2). \quad (11)$$

Setting $a^2 = b^2 + c^2$, (11) yields

$$(R^2 - q^2 - p^2 - c^2)^2 - 4(b^2 R^2 + p^2 c^2) = 0. \quad (12)$$

This is equivalent to the equation

$$\{R^2 - [q^2 + (p + c)^2]\} \cdot \{R^2 - [q^2 + (p - c)^2]\} = 4b^2 R^2, \quad (13)$$

and since

$$d_1^2 = q^2 + (p + c)^2, \quad d_2^2 = q^2 + (p - c)^2, \quad (14)$$

formula (2) holds.

We note that Theorem 2 can also be proved by means of analytic geometry, using Poncelet's porism.

Theorem 2 leads to the following result.

Theorem 3. *Let \mathfrak{C} be a circle about O with radius R and let \mathfrak{E} be an ellipse contained in \mathfrak{C} with semi-minor axis b and foci F_1, F_2 . Then, a necessary and sufficient condition for the existence of a triangle which includes \mathfrak{E} and is included in \mathfrak{C} is*

$$(R^2 - d_1^2)(R^2 - d_2^2) \geq 4b^2 R^2, \quad (15)$$

where $d_1 = \overline{OF_1}$, $d_2 = \overline{OF_2}$.

Proof. Consider the one-parameter family of confocal ellipses

$$\{\mathfrak{E}(t); \quad t \geq 0\}, \quad (16)$$

with semi-minor axis t and fixed foci F_1, F_2 . Our ellipse belongs to this family and we have $\mathfrak{E} = \mathfrak{E}(b)$.

Assume now the existence of a triangle $P_1 P_2 P_3$ which contains $\mathfrak{E}(b)$ and is contained in \mathfrak{C} . Then, by a perturbation argument, there exists a triangle with vertices $Q_1, Q_2 = Q_2(b)$ and $Q_3 = Q_3(b)$, which is inscribed in \mathfrak{C} and contains $\mathfrak{E}(b)$ such that two of its sides, $Q_1 Q_2(b)$ and $Q_1 Q_3(b)$, touch $\mathfrak{E}(b)$, as shown in Figure 1.

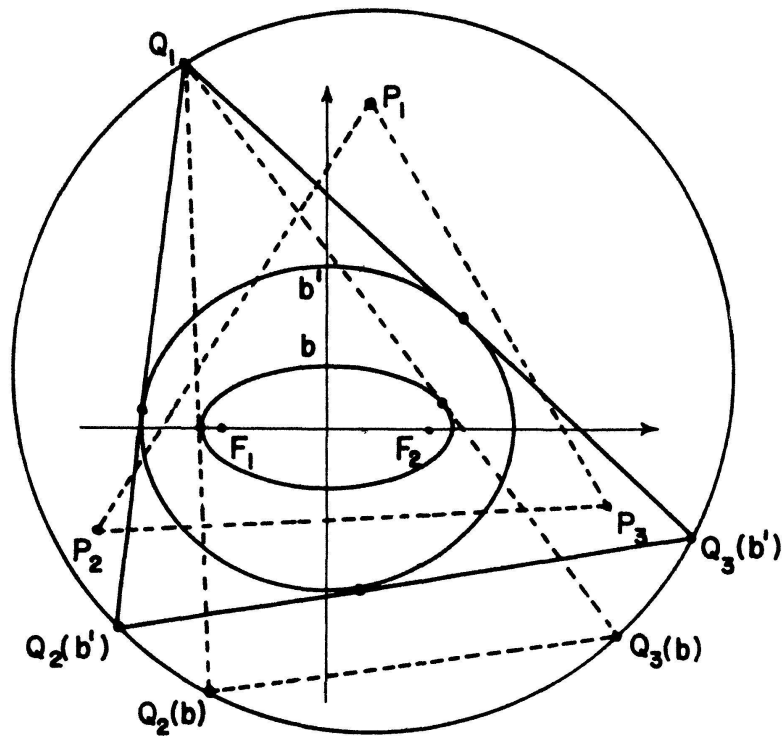


Figure 1

Now, keep Q_1 fixed. Starting with $t = b$, let t increase and let the points $Q_2 = Q_2(t)$ and $Q_3 = Q_3(t)$ move on the circle such that the sides $Q_1Q_2(t)$ and $Q_1Q_3(t)$ touch the ellipse $\mathfrak{C}(t)$. In this continuous process, the side $Q_2(t)Q_3(t)$ approaches $\mathfrak{C}(t)$, and for some $t = b'$ with $b' \geq b$, $Q_2(b')Q_3(b')$ touches $\mathfrak{C}(b')$. Hence, we have obtained a triangle inscribed in \mathfrak{C} and circumscribed about $\mathfrak{C}(b')$; the ellipse $\mathfrak{C}(b')$ being confocal with $\mathfrak{C}(b)$. By Theorem 2

$$(R^2 - d_1^2) (R^2 - d_2^2) = 4 b'^2 R^2, \tag{17}$$

and since $b' \geq b$, inequality (15) holds.

Conversely, assume that (15) holds and that there is no triangle which is included in \mathfrak{C} and includes $\mathfrak{C}(b)$. Then, by a similar argument as above, we decrease t to obtain a confocal ellipse $\mathfrak{C}(b')$ inscribed in a triangle which in turn is inscribed in \mathfrak{C} . Therefore, (17) is satisfied, and $b' < b$ implies

$$(R^2 - d_1^2) (R^2 - d_2^2) < 4 b^2 R^2. \tag{18}$$

This contradicts (15) and the theorem follows.

It seems interesting to derive metric relations analogous to those in Theorems 2 and 3, in the more general case of an ellipse within an ellipse.

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[1] D. M. Y. SOMMERVILLE, *Analytical Conics*, (G. Bell and Sons, LTD, London 1946).