

# On inscribed circumscribed conics

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## On Inscribed Circumscribed Conics

There is a well known geometric theorem due to Euler which states the following:

**Theorem 1** (Euler). *Given a triangle inscribed in a circle of radius  $R$  and circumscribed about a circle of radius  $r$ , then*

$$R^2 - d^2 = 2 r R, \quad (1)$$

where  $d$  is the distance between the circumcenter and the incenter of the triangle.

In this note first we generalize Euler's Theorem as follows:

**Theorem 2.** *Let  $\mathfrak{C}$  be a circle about  $O$  with radius  $R$  and let  $\mathfrak{E}$  be an ellipse contained in  $\mathfrak{C}$  with semi-minor axis  $b$  and foci  $F_1, F_2$ . Set  $d_1 = \overline{OF_1}$ ,  $d_2 = \overline{OF_2}$ . Then there exists a triangle inscribed in  $\mathfrak{C}$  and circumscribed about  $\mathfrak{E}$ , if and only if*

$$(R^2 - d_1^2)(R^2 - d_2^2) = 4 b^2 R^2. \quad (2)$$

*Proof.* Let  $\mathfrak{R}$  and  $\mathfrak{Q}$  be two conics in the projective plane, given in homogeneous coordinates by

$$X^t A X = 0 \quad \text{and} \quad X^t B X = 0 \quad (3)$$

respectively, so that  $\mathfrak{R}$  contains  $\mathfrak{Q}$ . It is known (cf. [1] p. 279) that a necessary and sufficient condition for the existence of a triangle inscribed in  $\mathfrak{R}$  and circumscribed about  $\mathfrak{Q}$  is

$$\theta^2 = 4 \Delta \theta', \quad (4)$$

where  $\Delta$ ,  $\theta$ ,  $\theta'$ , (and  $\Delta'$ ), are determined by

$$\det(A + \lambda B) \equiv \Delta + \theta \lambda + \theta' \lambda^2 + \Delta' \lambda^3. \quad (5)$$

It can be shown that

$$\Delta = \det A, \quad \theta = \text{tr}[(\text{adj } A)B], \quad \theta' = \text{tr}[(\text{adj } B)A], \quad (\Delta' = \det B), \quad (6)$$

where  $\text{tr}$  denotes the trace and  $\text{adj}$  the matrix of cofactors. Thus, the condition in (4) takes the form

$$\text{tr}^2[(\text{adj } A)B] = 4 \det A \cdot \text{tr}[(\text{adj } B)A]. \quad (7)$$

In our case let the ellipse and the circle be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (8)$$

and

$$(x - p)^2 + (y - q)^2 = R^2, \quad (9)$$

respectively. Then the non-singular symmetric matrices in (7) are

$$A = \begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -p \\ 0 & 1 & -q \\ -p & -q & p^2 + q^2 - R^2 \end{pmatrix}. \quad (10)$$

Therefore, in this particular case, (7) reads

$$(R^2 + a^2 + b^2 - p^2 - q^2)^2 = 4(a^2 b^2 - a^2 q^2 - b^2 p^2 + a^2 R^2 + b^2 R^2). \quad (11)$$

Setting  $a^2 = b^2 + c^2$ , (11) yields

$$(R^2 - q^2 - p^2 - c^2)^2 - 4(b^2 R^2 + p^2 c^2) = 0. \quad (12)$$

This is equivalent to the equation

$$\{R^2 - [q^2 + (p + c)^2]\} \cdot \{R^2 - [q^2 + (p - c)^2]\} = 4b^2 R^2, \quad (13)$$

and since

$$d_1^2 = q^2 + (p + c)^2, \quad d_2^2 = q^2 + (p - c)^2, \quad (14)$$

formula (2) holds.

We note that Theorem 2 can also be proved by means of analytic geometry, using Poncelet's porism.

Theorem 2 leads to the following result.

**Theorem 3.** *Let  $\mathfrak{C}$  be a circle about  $O$  with radius  $R$  and let  $\mathfrak{E}$  be an ellipse contained in  $\mathfrak{C}$  with semi-minor axis  $b$  and foci  $F_1, F_2$ . Then, a necessary and sufficient condition for the existence of a triangle which includes  $\mathfrak{E}$  and is included in  $\mathfrak{C}$  is*

$$(R^2 - d_1^2)(R^2 - d_2^2) \geq 4b^2 R^2, \quad (15)$$

where  $d_1 = \overline{OF_1}$ ,  $d_2 = \overline{OF_2}$ .

*Proof.* Consider the one-parameter family of confocal ellipses

$$\{\mathfrak{E}(t); \quad t \geq 0\}, \quad (16)$$

with semi-minor axis  $t$  and fixed foci  $F_1, F_2$ . Our ellipse belongs to this family and we have  $\mathfrak{E} = \mathfrak{E}(b)$ .

Assume now the existence of a triangle  $P_1 P_2 P_3$  which contains  $\mathfrak{E}(b)$  and is contained in  $\mathfrak{C}$ . Then, by a perturbation argument, there exists a triangle with vertices  $Q_1, Q_2 = Q_2(b)$  and  $Q_3 = Q_3(b)$ , which is inscribed in  $\mathfrak{C}$  and contains  $\mathfrak{E}(b)$  such that two of its sides,  $Q_1 Q_2(b)$  and  $Q_1 Q_3(b)$ , touch  $\mathfrak{E}(b)$ , as shown in Figure 1.

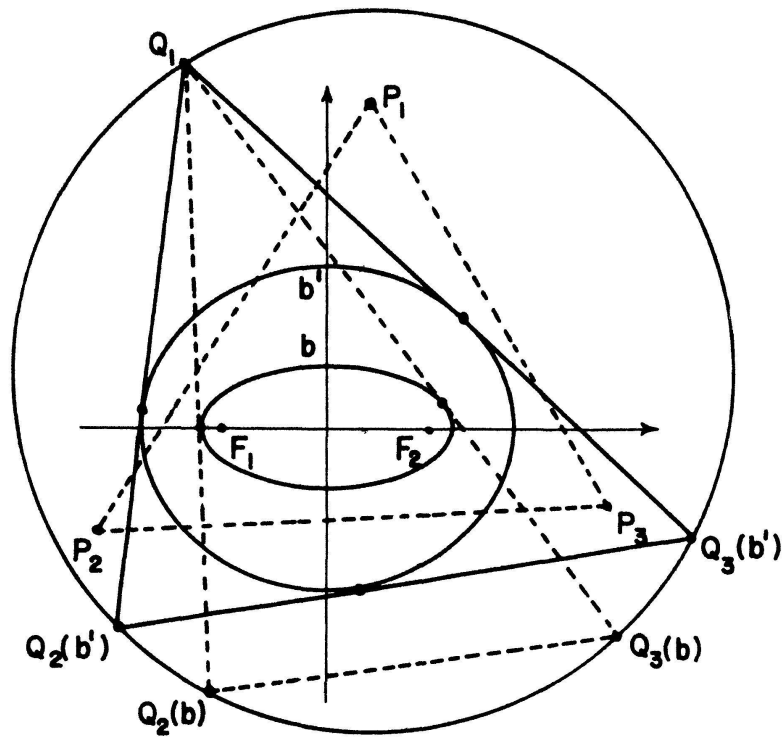


Figure 1

Now, keep  $Q_1$  fixed. Starting with  $t = b$ , let  $t$  increase and let the points  $Q_2 = Q_2(t)$  and  $Q_3 = Q_3(t)$  move on the circle such that the sides  $Q_1Q_2(t)$  and  $Q_1Q_3(t)$  touch the ellipse  $\mathfrak{C}(t)$ . In this continuous process, the side  $Q_2(t)Q_3(t)$  approaches  $\mathfrak{C}(t)$ , and for some  $t = b'$  with  $b' \geq b$ ,  $Q_2(b')Q_3(b')$  touches  $\mathfrak{C}(b')$ . Hence, we have obtained a triangle inscribed in  $\mathfrak{C}$  and circumscribed about  $\mathfrak{C}(b')$ ; the ellipse  $\mathfrak{C}(b')$  being confocal with  $\mathfrak{C}(b)$ . By Theorem 2

$$(R^2 - d_1^2) (R^2 - d_2^2) = 4 b'^2 R^2, \tag{17}$$

and since  $b' \geq b$ , inequality (15) holds.

Conversely, assume that (15) holds and that there is no triangle which is included in  $\mathfrak{C}$  and includes  $\mathfrak{C}(b)$ . Then, by a similar argument as above, we decrease  $t$  to obtain a confocal ellipse  $\mathfrak{C}(b')$  inscribed in a triangle which in turn is inscribed in  $\mathfrak{C}$ . Therefore, (17) is satisfied, and  $b' < b$  implies

$$(R^2 - d_1^2) (R^2 - d_2^2) < 4 b^2 R^2. \tag{18}$$

This contradicts (15) and the theorem follows.

It seems interesting to derive metric relations analogous to those in Theorems 2 and 3, in the more general case of an ellipse within an ellipse.

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