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Kleine Mitteilungen

A triangle inequality

Aufgabe 743 of this journal, proposed by J. Paasche, reads as follows. A chain of inequalities is given:

$$0 \leq \frac{Rr-2r^2}{1} \leq \frac{s^2-27r^2}{A} \leq \frac{2s^2-27Rr}{B} \leq \frac{R^2-2Rr}{C}$$

$$\leq \frac{R^2-4r^2}{D} \leq \frac{27R^2-4s^2}{E}.$$

Determine successively the maximum value of the positive numbers A, B, C, D, E such that the inequalities hold for any triangle; s is its semi-perimeter, R and r are the radii of the circumcircle and the inscribed circle.
A. Bager was the only one who sent in a complete solution; it has been published in volume 31, No. 3, p. 67–70, Mai 1976. For A, B, C, D the problem is by no means trivial, but it is simple in comparison with that for E. Bager finds

$$A=16, \qquad B=5, \qquad C=D=\frac{5}{8},$$

and, after an ingenious argumentation,

$$E=\frac{20+5\sqrt{17}}{8}.$$

Some time ago we developed a method to derive inequalities involving s, R and r in a systematic way [1]. By this method Bager's results are completely confirmed. If the basic idea of the procedure is accepted the proofs are relatively simple and they could have the advantage to visualize more or less the situation. We introduce $x=r/R$, $y=s/R$ and map any triangle (or, more precisely, any class of similar triangles) on the point (x,y) of a cartesian frame. It can be proved that the set of image points is the region G (Fig. 1), bordered by the arcs OA_1 and A_1D of a deltoid (or Steiner's hypocycloid) d and the segment OD of the Y-axis; $A_1=(1/2, (3\sqrt{3})/2)$, $D=(0,2)$. Points on OA_1 or on A_1D are the images of isosceles triangles, A_1 corresponds to the equilateral triangle; points on OD correspond to degenerated triangles.
It is clear that $K(x,y) \geq 0$ or $K(x,y) \leq 0$ is an inequality (involving s, R, r) which holds for every triangle if and only if no internal points of G are on the curve $K=0$. We could determine the maximum values of A, B, C, D in this way, but we shall restrict ourselves to E, supposing $D=5/8$.

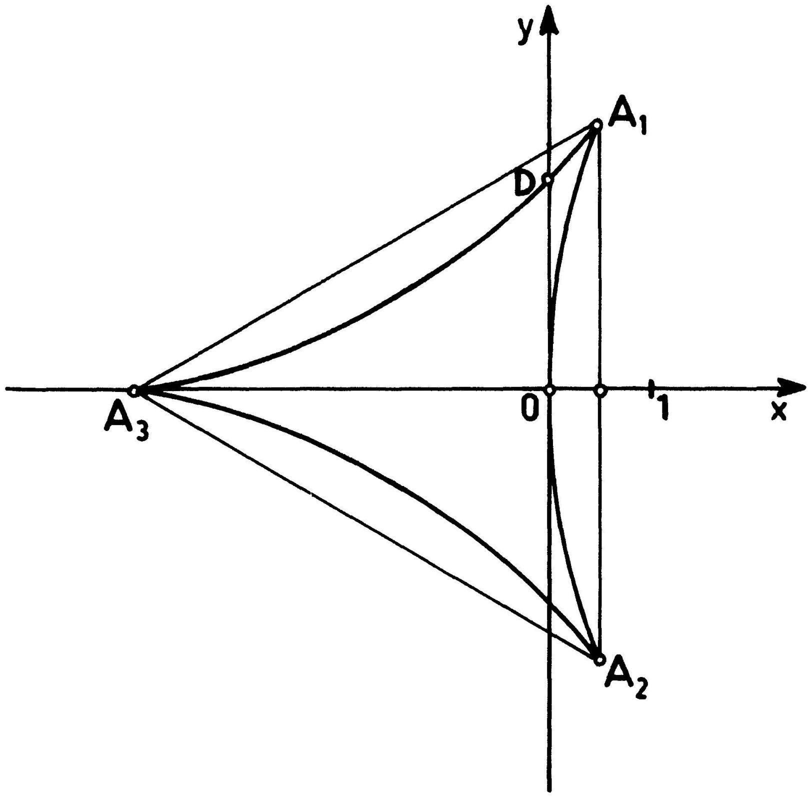

Figure 1

Putting $\lambda=(8/5)E$, it comes to this: we ask for the maximum value of λ such that for any point of G the inequality

$$K \equiv \lambda(1-4x^2)-(27-4y^2) \leq 0 \tag{1}$$

holds.

$K=0$, for variable λ, represents a pencil of conics; all conics have OX and OY as axes of symmetry and they pass through A_1. As $E>0$ we have $\lambda>0$, which means that we have only to consider the hyperbolas of the pencil. Moreover, as for instance $D=(0, 2)$ must satisfy (1) we have $\lambda \leq 11 < 27$, which implies that the hyperbolas intersect OY at real points. Our condition is therefore: the upper branch of K should not penetrate into G; that means that K and the arc A_1D have no real and different intersections (Fig. 2).

A parameter representation of d reads

$$x=\frac{4t^2(1-t^2)}{(1+t^2)^2}, \qquad y=\frac{8t}{(1+t^2)^2}, \tag{2}$$

where O, A_1 and D correspond to $t=0$, $t=(\sqrt{3})/3$ and $t=1$. The intersections of K and d follow if we substitute (2) into (1). We obtain a quartic equation for $t^2=u$, which could be expected because K and d have both OX as an axis of symmetry; moreover $t=(\sqrt{3})/3$, $u=1/3$ must be a double root for A_1 is a double point of d. We obtain

$$1-4x^2=(1+u)^{-4}\{(1+u)^4-64u^2(1-u)^2\}$$
$$=(1+u)^{-4}(3u-1)^2(-7u^2+10u+1), \tag{3}$$

and

$$27-4y^2=(1+u)^{-4}\{27(1+u)^4-256u\}$$
$$=(1+u)^{-4}(3u-1)^2(3u^2+14u+27). \tag{4}$$

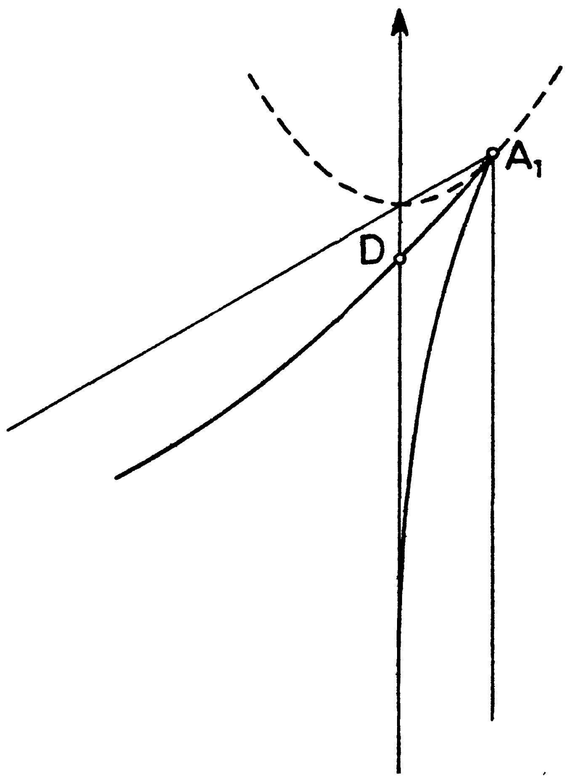

Figure 2

Hence the intersections of K and d, different from the double points, follow from

$$(7\lambda+3)u^2+(-10\lambda+14)u+(-\lambda+27)=0, \tag{5}$$

with the discriminant

$$\delta=32(\lambda^2-8\lambda-1), \tag{6}$$

which implies that the maximum value of λ such that K does not penetrate into G is equal to $4+\sqrt{17}$, which gives us indeed $E=(20+5\sqrt{17})/8$. In this case K is tangent to d at the point $u=(-3+\sqrt{17})/2$, which is a point between A_1 and D. It is the image of the isosceles triangle for which, apart from the equilateral triangle, the equality sign holds.

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REFERENCE

1 O. Bottema: Inequalities for R, r and s. Publ. Elektr. Fak. Univ. Beogradu, No. 340, p. 27–36 (1971).