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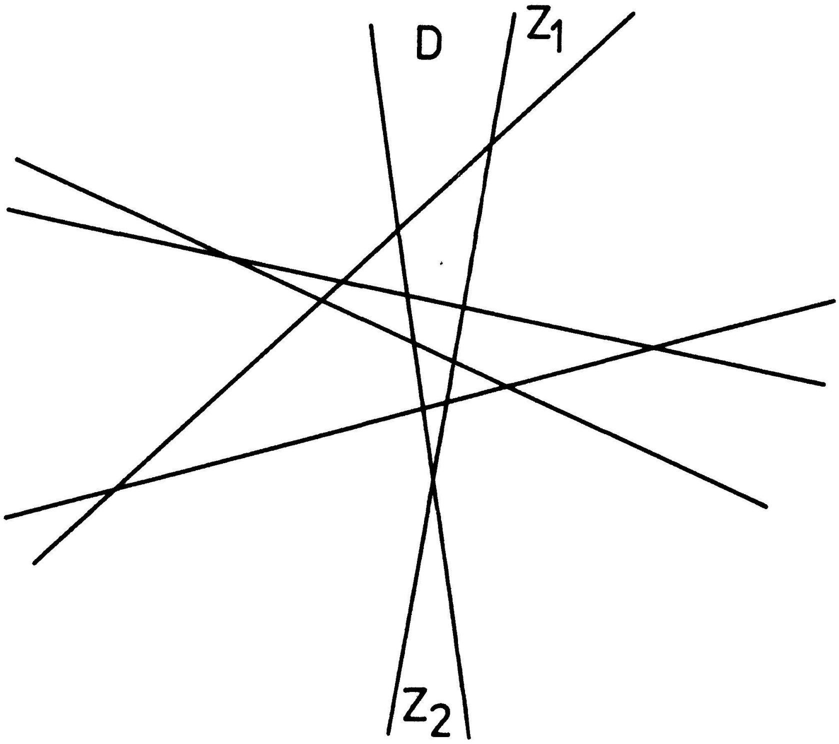


Abb. 2. Eine Anordnung $A_{V,E}$ (6).

struierten einfachen Anordnung $A_{V,E}(n)$ folgt etwa aus den verschiedenen Schnittpunktanzahlen.

Damit ist Satz 3 bewiesen. Es kann ausserdem gezeigt werden, dass für $n \equiv 0 \pmod 4$ weitere maximale Anordnungen $A_{V,E}(n)$ existieren, und auch für $n \not\equiv 0 \pmod 4$ gibt es, bis auf einige Ausnahmen, nichtisomorphe Anordnungen $A_{V,E}(n)$.

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LITERATURVERZEICHNIS

1 B. Grünbaum: Arrangements und spreads. Am. Math. Soc., Providence, R.I. (1972).

2 H. Harborth und I. Mengersen: Geradenanordnungen mit maximaler Anzahl vierseitiger Flächen (eingereicht).

# Kleine Mitteilungen

## An application of Dirichlet convolution in proving some inequalities from elementary number theory

1. The aim of this note is to show how we can use the properties of a ring of real arithmetic functions to prove some known inequalities from the elementary theory of numbers.

We denote by $f$ an arithmetic function and by $f(n)$ the value of $f$ at $n$. Let $A$ be the set of all real arithmetic functions. If $f, g \in A$, then their Dirichlet convolution is defined by

$$(f*g)(n)=\sum_{k|n} f(k)g(n/k), \quad n \geqslant 1.$$

It is well known that $A$ forms a commutative ring with convolution as multiplication and with the usual addition. This ring has the zero and unit element $\Theta$ and $e$, respectively, defined by

$$\Theta(n)=0, \quad n \geqslant 1; \qquad e(n)=\begin{cases}1 & \text{for } n=1\\ 0 & \text{for } n \neq 1.\end{cases}$$

We further define the product $fg$ as usual,

$$(fg)(n)=f(n)g(n), \quad n \geqslant 1.$$

We write $f \leqslant g$ $(f \geqslant g)$ when $f(n) \leqslant g(n)$ $(f(n) \geqslant g(n))$ for $n \geqslant 1$. We use in the proofs of all inequalities under consideration the following trivial lemma:

**Lemma.** *If $f, g \in A$ and $f \geqslant \Theta$, $g \geqslant \Theta$, then $f * g \geqslant \Theta$.*

2. We shall denote by $d(n)$ and $\sigma_k(n)$ the number of divisors and the sum of $k$-th powers of divisors of $n$ respectively $(\sigma_1(n)=\sigma(n))$, by $\mu$ and $\varphi_k$ the Möbius function and the Jordan function, respectively $(\varphi_k(n)=n^k \Pi_{p|n}(1-p^{-k})$, $p$ prime, is defined for positive integers $k, n$ as the number of different sequences $a_1, \ldots, a_k$ containing $k$ (equal or distinct) positive integers $\leqslant n$ such that $(a_1, \ldots, a_k, n)=1$; $\varphi_1=\varphi$ is the Euler function$)$. For integral $k$, define the arithmetic function $I_k$ by

$$I_k(n)=n^k, \quad n \geqslant 1$$

$(I_0=I)$. In the proofs we shall use the following known equalities valid in the ring $A$:

$$d=I*I, \quad (1) \qquad I*\mu=e, \quad (2) \qquad I*\varphi_k=I_k, \quad (3)$$
$$\varphi_k I_{-k}=I*(\mu I_{-k}), \quad (4) \qquad \sigma_k I_{-k}=I*I_{-k}, \quad (5) \qquad \sigma_k=\varphi_k*d. \quad (6)$$

The first inequality is due to Bagchi and Gupta [1]:

2.1. If $n>1$ then $\sigma(n) \geqslant \varphi(n)+d(n)$.

Proof: It suffices to show that $\sigma+e \geqslant \varphi+d$, that is $\sigma+e-\varphi-d \geqslant \Theta$. From (6) for $k=1$ we obtain

$$\sigma+e-\varphi-d=\varphi*d+e-\varphi-d=(\varphi-e)*(d-e).$$

Since $\varphi-e \geqslant \Theta$ and $d-e \geqslant \Theta$ the inequality $(\varphi-e)*(d-e) \geqslant \Theta$ holds by the lemma and the proof of (2.1) is completed.

Remarks: It is easy to notice that $((\varphi - e) * (d - e))\ (n) = 0$ holds iff $n = 1$ or $n$ is a prime number. Hence $\sigma(n) > \varphi(n) + d(n)$ iff $n$ is a composite number. We can get the inequality (2.1) from the relation $\sigma = \varphi * d$ also as follows: if $n > 1$ then

$$\sigma(n) = \sum_{k \mid n} \varphi(k)\, d(n/k) \geqslant \varphi(1)\, d(n) + \varphi(n)\, d(1) = \varphi(n) + d(n).$$

The second inequality is due to Makowski [2]:

2.2. For $n, k \geqslant 1$ we have $\varphi_k(n) + \sigma_k(n) \geqslant 2n^k$.

Proof: From the relations (6) and (3) we have

$$\varphi_k + \sigma_k - 2I_k = \varphi_k + (\varphi_k * d) - (\varphi_k * 2I) = \varphi_k * (e + d - 2I).$$

Applying the lemma we obtain inequality (2.2) immediately.

Remark: As (2.1) it is easy to verify that the strong inequality $\varphi_k(n) + \sigma_k(n) > 2n^k$ holds iff $n$ is a composite number.

The third inequality is due to Makowski [2]:

2.3. If $k \geqslant 1$, $n > 1$ then $\varphi_k(n) + \sigma_k(n) \leqslant n^k d(n)$.

Proof: If suffices to prove the inequality

$$\varphi_k I_{-k} + \sigma_k I_{-k} \leqslant d + e.$$

Using the relations (1), (2), (4), (5) and the lemma we have

$$d + e - \sigma_k I_{-k} - \varphi_k I_{-k} = I * \left((I + \mu) - (I + \mu) I_{-k}\right) = I * \left((I + \mu)(I - I_{-k})\right) \geqslant \Theta$$

and (2.3) follows.

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REFERENCES

1 H. das Bagchi and M. Gupta: Problem 343. Jber. Dt. Math. Verein. *57*, 8–9 (italics) (1954).
2 A. Makowski: Problem 339. El. Math. *15*, 39–40 (1960).

# Aufgaben

**Aufgabe 804.** Man bestimme die Anzahl der inkongruenten ebenen Netze eines regulären Ikosaeders. [Vgl. M. Jeger: Über die Anzahl der inkongruenten ebenen