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inequalities, lies in the choice of the constraint set. The relations between the elements of a triangle are often given in terms of circle functions. These functions, when appearing in the constraint functions, greatly complicate the determination of points satisfying the first order conditions.

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Distance theorems in geometry

1. Introduction

The purpose of this note is to give a method for proving 'distance theorems' in elementary plane geometry. As an application we give an easy proof of the Feuerbach theorem and we solve an old problem of A. H. Stone [3] problem E585.
Let (T) be any triangle $A_1A_2A_3$ with vertices numbered in counter clockwise order. Denote the interior angle at A_i by $\alpha_i (i=1,2,3)$, and the length of the opposite side by a_i. We use the notation $P(x_1, x_2, x_3)$ or (x_i) to indicate that the distances of P from the sides of (T) are proportional to x_1, x_2, x_3 with the convention that x_i is positive if P and A_i are on the same side of a_i and negative otherwise. We shall also use capital letters to denote complex numbers; thus, for example, $(1/3)(A_1+A_2+A_3)$ is the centroid of (T).
Our method is based on the following elementary lemma.

Lemma. *Let M be a point in the plane of (T) satisfying*

$$\sum_{i=1}^{3} m_i \overline{MA}_i^2 = k, \tag{1}$$

where the m_i's are real numbers satisfying $s_3 = m_1 + m_2 + m_3 \neq 0$, and k is a constant satisfying

$$s_3 k-(m_1 m_2 a_3^2+m_2 m_3 a_1^2+m_3 m_1 a_2^2)\geq 0\,. \tag{2}$$

Then the locus of M is a circle $C(A;\,r)$ with center A given by

$$A\,(m_1/a_1,\, m_2/a_2,\, m_3/a_3) \tag{3}$$

and radius r given by

$$s_3^2 r^2=s_3 k-(m_1 m_2 a_3^2+m_2 m_3 a_1^2+m_3 m_1 a_2^2)\,. \tag{4}$$

Conversely, any circle in the plane of (T) with center $A\,(x_1, x_2, x_3)$ and radius r, is the locus of points M satisfying (1) *with $m_i=a_i x_i$ and k given by* (4).

The number $m_i\,(i=1,2,3)$ will be called the *weight* at A_i.
Note that in the converse, the weights m_1, m_2, m_3 and the constant k are not unique.

Proof: Let M be a point in the plane of (T) satisfying (1) and (2) and let A be the point $(1/s_3)\,(m_1 A_1+m_2 A_2+m_3 A_3)$.
Since $s_3=m_1+m_2+m_3\neq 0$, the sum of at least two of the weights is not zero. Without loss of generality we may assume that $s_2=m_1+m_2\neq 0$. Writing

$$A=\frac{1}{3}(m_1 A_1+m_2 A_2+m_3 A_3)=\frac{s_2}{s_3}\,\frac{(m_1 A_1+m_2 A_2)}{s_2}+\frac{m_3}{s_3}A_3$$

we, see immediately that the distances of the point A from the sides of (T) are proportional to $m_1/a_1, m_2/a_2, m_3/a_3$.
We also have,

$$\sum_{i=1}^{3} m_i \overline{MA}_i^2=\left(\sum_{i=1}^{3} m_i\right)\overline{MA}^2+\frac{1}{s_3}(m_1 m_2 a_3^2+m_2 m_3 a_1^2+m_3 m_1 a_2^2) \tag{5}$$

which together with (2) shows that the distance $\overline{MA}$ between M and A is constant and non-negative. Hence the locus of M is a circle $C(A;\,r)$ with center A given by (3) and radius r given by (4).

Conversely, if $C(A;\,r)$ is a circle in the plane of (T) with center $A\,(x_i)$ and radius r, then, putting $m_i=a_i x_i$ we see that $A=(m_i/a_i)$ and, from (5), $k=\sum m_i \overline{MA}_i^2$ is given by (4).

Examples:

Circle	Center	Radius	Weight at A_i	k
Circumcircle	$O(\cos a_i)$	R	$a_i \cos a_i$	$a_1 a_2 a_3$
Incircle	$I(1,1,1)$	ρ	a_i	$2\Delta\rho+a_1 a_2 a_3$
Excircle in a_1	$I_1(-1,1,1)$	ρ_1	$(-a_1, a_2, a_3)$	$2\Delta\rho_1-a_1 a_2 a_3$

Δ = area of (T).

2. The method

Let $P(x_i)$ and $Q(y_i)$ be two points in the plane of (T). To find the distance between P and Q we find the radius of the circle centered at P and passing through Q. Such a circle has weights $m_i = a_i x_i$ and corresponding constant k equal to $\sum m_i \overline{QA}_i^2 = \sum a_i x_i \overline{QA}_i^2$. Now the distance between P and Q is obtained by solving for r in equation (4).

An important special case occurs when Q is equidistant from the vertices of (T), i.e. when $Q = O$. In this case the distance d between O and P will be given by

$$d^2 = R^2 - \frac{a_1 a_2 a_3}{s_3^2}(x_1 x_2 a_3 + x_2 x_3 a_1 + x_3 x_1 a_2) \tag{6}$$

where

$$s_3 = \sum_{i=1}^{3} a_i x_i .$$

Examples:

1. The distance d between the incenter I and the circumcenter O of (T) is given by

$$d^2 = R^2 - \frac{a_1 a_2 a_3}{a_1 + a_2 + a_3} = R^2 - 2R\rho .$$

This is Euler's theorem ([2], p. 186, and [1], p. 85).

2. The centroid G of (T) has coordinates $(1/a_1, 1/a_2, 1/a_3)$.
The distance OG is given by

$$\overline{OG}^2 = R^2 - \frac{1}{9}(a_1^2 + a_2^2 + a_3^2) .$$

Theorem *(Feuerbach). The nine-point circle of a triangle touches the incircle and each of the excircles.*

Proof: Denote the nine-point center of (T) by N, its orthocenter by H, and retain the notations of the previous section. We have to show that the distance between the incenter I and N is equal to the difference of the radii of the incircle and the nine-point circle.

Since N is the midpoint of OH, the median formula may be applied in triangle HA_iO to give:

$$\overline{NA}_i^2 = \frac{1}{2}\left(\overline{HA}_i^2 + \overline{OA}_i^2 - \frac{1}{2}\overline{OH}^2\right) = \frac{1}{2}\left[(2R\cos a_i)^2 + R^2 - \frac{1}{2}(3\,\overline{OG})^2\right]$$

$$= \frac{1}{4}(R^2 - 8R^2 \sin^2 a_i + a_1^2 + a_2^2 + a_3^2)$$

$$= \frac{1}{4}(R^2 + 4\Delta \cot a_i) . \tag{7}$$

For the circle centered at I and passing through N, the weights are a_1, a_2, a_3 and the constant k is

$$\sum a_i \overline{NA}_i^2 = \frac{1}{4} R^2 (a_1+a_2+a_3) + 2R\Delta \sum \cos a_i$$
$$= \frac{1}{4} R^2 (a_1+a_2+a_3) + 2R\Delta\left(\frac{\rho}{R}+1\right).$$

Using this value for k in (4) and solving for r we get

$$r^2 = \overline{IN}^2 = \frac{1}{\sum a_i}(k - a_1 a_2 a_3) = \frac{\rho}{2}(k - 4R\Delta)$$
$$= \frac{\rho}{2\Delta}\cdot(R^2 - 4\rho^2 - 4R\rho)\cdot\frac{\Delta}{2\rho} = \left(\frac{R}{2}-\rho\right)^2. \tag{8}$$

Since the radius of the nine-point circle is $R/2$, (8) shows that the incircle and the nine-point circle are internally tangent.

For the circle centered at I_I and passing through N the constant k is

$$k = \frac{1}{4} R^2 (a_2+a_3-a_1) + 2R(\cos a_2 + \cos a_3 - \cos a_1)$$
$$= \frac{1}{4} R^2 (a_2+a_3-a_1) + 2R\Delta\left(\frac{\rho_1}{R}-1\right), \tag{9}$$

and we find as before, that $\overline{I_1 N}^2 = (R/2+\rho_1)^2$. Thus the excircle of center I_1 is externally tangent to the nine point circle.

We now deal with an old problem of A.H. Stone ([3], p. 1026): Let the circle of center A and radius R cut the sides a_1, a_2 and a_3 of triangle $A_1A_2A_3$ in the points X, X', Y, Y' and Z, Z', respectively. Let M and M' be the Miquel points of the triangles XYZ and $X'Y'Z'$, respectively.

Theorem. *M and M' are equidistant from A.*

Proof: First observe that formula (6) can be written in the form

$$d^2 = R^2\left\{1 - \frac{4\sin a_1 \sin a_2 \sin a_3 (x_1x_2\sin a_3 + x_2x_3\sin a_1 + x_3x_1\sin a_2)}{(\sum x_i \sin a_i)^2}\right\}. \tag{10}$$

Let (u_1, u_2, u_3) be trilinear coordinates of M with respect to XYZ as triangle of reference and let (v_1, v_2, v_3) be trilinear coordinates of M' with respect to $X'Y'Z'$ as triangle of reference. Let x, y and z denote the measures of the angles of XYZ and let x', y' and z' denote the measures of the angles of $X'Y'Z'$. Since XYZ and $X'Y'Z'$ have the same circumcircle, (10) shows that $\overline{MA}^2 = \overline{M'A}^2$ if and only if

$$\frac{\sin x \sin y \sin z\,(u_1u_2\sin z + u_2u_3\sin x + u_3u_1\sin y)}{(u_1\sin x + u_2\sin y + u_3\sin z)^2}$$
$$= \frac{\sin x' \sin y' \sin z'\,(v_1v_2\sin z' + v_2v_3\sin x' + v_3v_1\sin y')}{(v_1\sin x' + v_2\sin y' + v_3\sin z')^2}. \tag{11}$$

The rest of the proof consists in showing that (11) holds true. We have to compute (u_1, u_2, u_3) and (v_1, v_2, v_3). Inscribe in triangle XYZ a triangle $B_1B_2B_3$ similar to $A_1A_2A_3$ and similarly placed, e.g. B_1B_3 parallel to A_1A_3, etc.

If P is the Miquel point of $B_1B_2B_3$ with respect to XYZ then P has coordinates

$$\frac{\sin(x+a_1)}{\sin a_1}, \quad \frac{\sin(y+a_2)}{\sin a_2}, \quad \frac{\sin(z+a_3)}{\sin a_3}$$

with respect to XYZ as triangle of reference. Also [2], p. 271, P is isogonal conjugate to M and so M has coordinates

$$\frac{\sin a_1}{\sin(x+a_1)}, \quad \frac{\sin a_2}{\sin(y+a_2)}, \quad \frac{\sin a_3}{\sin(z+a_3)} \tag{12}$$

with respect to XYZ.
Similarly M' has coordinates

$$\frac{\sin a_1}{\sin(x'+a_1)}, \quad \frac{\sin a_2}{\sin(y'+a_2)}, \quad \frac{\sin a_3}{\sin(z'+a_3)} \tag{13}$$

with respect to $X'Y'Z'$.

Using theorem 239, [2], p. 157, we have (say when X, X' are between A_2 and A_3, Y, Y' are between A_3A_1, etc.).

$$x+x'=a_2+a_3, \qquad y+y'=a_3+a_1, \qquad z+z'=a_1+a_2, \tag{14}$$

from which follows easily that $\sin(z'+a_3)=\sin z$, $\sin(z+a_3)=\sin z'$, etc. or

$$\sin a_1 \sin a_2 \sin z \sin(z+a_3)=\sin a_1 \sin a_2 \sin z' \sin(z'+a_3), \text{ etc.} \tag{15}$$

Also

$$\begin{aligned}
&\sin a_1 \sin x \sin(a_2+y)\sin(a_3+z)+\sin a_2 \sin y \sin(a_3+z)\sin(a_1+x)\\
&=\sin(a_3+z)\{\sin a_3 \sin y \sin x+\sin a_1 \sin a_2 \sin(x+y)\}\\
&=\sin a_3 \sin z' \sin(a_1+x')\sin(a_2+y')+\sin a_1 \sin a_2 \sin(a_3+z)\sin z\\
&=\sin a_3 \sin z' \sin(a_1+x')\sin(a_2+y')+\sin a_1 \sin a_2 \sin z' \sin(a_3+z')\\
&=\sin a_3 \sin z' \sin(a_1+x')\sin(a_2+y')+\sin a_1 \sin a_2 \sin(x'+y')\sin(a_3+z')\\
&\quad+\{\sin a_1 \sin x' \sin(a_2+y')\sin(a_3+z')+\sin a_2 \sin y' \sin(a_3+z')\sin(a_1+x')\\
&\quad-\sin(a_3+z')\sin a_3 \sin y' \sin x'\}.
\end{aligned} \tag{16}$$

The last term in the braces is, by (14), $\sin z \sin a_3 \sin(a_2+y)\sin(a_1+x)$. Combining (16) and (15) it is now a simple matter to verify that (11) holds true.

Corollary. *The Brocard points of $A_1A_2A_3$ are equidistant from K, the Lemoine point of $A_1A_2A_3$.*

Proof: The cosine circle, whose center is K, cuts the sides of $A_1A_2A_3$ in the six points P_1, Q_1, P_2, Q_2, P_3, Q_3. The triangles $P_2P_3P_1$ and $Q_3Q_1Q_2$ have Ω and Ω' (the Brocard points) as their Miquel points ([2], p. 271).

4. Remarks

a) One can show by induction that if A_1, $A_2, \ldots, A_n$ are n points in space with weights $m_1, m_2, \ldots, m_n$ such that $s_n = \sum m_i \neq 0$ then for any point M in space

$$\sum_{i=1}^{n} m_i \overline{MA}_i^2 = s_n \overline{MA}^2 + \frac{1}{s_n} \sum_{1 \leq i < j \leq n} m_i m_j \overline{A_i A}_j^2 \tag{17}$$

where (with the appropriate convention)

$$A = \frac{1}{s_n}(m_1 A_1 + m_2 A_2 + \cdots + m_n A_n).$$

This makes it possible to extend our method to higher dimensions.

b) The 'continuous' analogue of (17) is

$$\left[\int_a^b p(x)\,dx\right]\left[\int_a^b p(x) f^2(x)\,dx\right] = \left[\int_a^b p(x) f(x)\,dx\right]^2$$

$$+ \iint_{a \leq u < v < b} p(u)\,p(v)\,[f(v) - f(u)]^2\,du\,dv$$

provided that the various integrals exist, and,

$$\int_a^b p(x)\,dx \neq 0.$$

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Kleine Mitteilungen

Kreispackung in Quadraten

1. Diese kurze Arbeit behandelt ein Problem in der euklidischen Ebene. Sei E im folgenden ein abgeschlossenes Einheitsquadrat mit den Ecken A, B, C, D; sei $\mathring{E}$ der offene Kern von E. Man greife k paarweise verschiedene Punkte P_i $(1 \leq i \leq k)$ aus E heraus und bestimme ihren Mindestabstand d. Das Quadrat $G(d)$ mit Seitenlänge $1+d$ gehe aus $\mathring{E}$ durch eine Streckung hervor, welche als Zentrum den Mittelpunkt von E besitzt. Die k offenen Kreise mit Mittelpunkt P_i und Radius $d/2$ unterdecken $G(d)$; die Dichte dieser Unterdeckung ist $(k\pi d^2)/4(1+d)^2$. Unter allen Möglich-