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Diophantine representation of generalized Fibonacci numbers

Let A and B be integers such that either $A>0$ and $B=-1$ or $A>3$ and $B=1$. We define a sequence $R=\{R_n\}_{n=0}^{\infty}$ by the integers $R_0=0$, $R_1=1$ and the recurrence

$$R_n=AR_{n-1}-BR_{n-2}, \qquad n>1.$$

When $A=-B=1$ the positive terms of sequence R are the Fibonacci numbers; when $A=-B=1$, $R_0=2$, $R_1=1$ they are the Lucas numbers.
Jones [1, 2] has proved that the set of all Fibonacci (respectively Lucas) numbers is identical with the set of positive values of the polynomial

$$y(2-(y^2-yx-x^2)^2),$$

respectively

$$y(1-((y^2-yx-x^2)^2-25)^2),$$

as the variables x and y range over the positive integers.
The purpose of this paper is to extend the results of Jones on Fibonacci numbers to the generalized sequence. We prove two theorems and a lemma.

Theorem 1. *For non-negative integers x, y*

$$|x^2-Axy+By^2|=1 \tag{1}$$

if and only if x and y are consecutive terms of sequence R.

Theorem 2. *The set of all terms of sequence R is identical with the set of all non-negative values of the polynomial*

$$f(x,y)=y(2-(x^2-Axy+By^2)^2)$$

as the variables x and y range over the non-negative integers.

Lemma. *For every non-negative integer n,*

$$|R_{n+1}^2 - AR_{n+1}R_n + BR_n^2| = 1.$$

Proof of the lemma: The lemma is obviously true for $n=0$. And by definition of sequence R,

$$|R_{n+1}^2 - AR_{n+1}R_n + BR_n^2| = |(AR_n - BR_{n-1})^2 - A(AR_n - BR_{n-1})R_n + BR_n^2|$$
$$= |B(R_n^2 - AR_nR_{n-1} + BR_{n-1}^2)| = |R_n^2 - AR_nR_{n-1} + BR_{n-1}^2|$$

for $n>0$, since $|B|=1$.

Proof of theorem 1: Equality (1) holds for $x=R_{n+1}$, $y=R_n$ by the lemma. Thus we must prove that if (1) holds for integers $x,y \geqq 0$, then x and y are consecutive terms of sequence R.

Suppose that for integers $x_0, y_0 > 0$ we have

$$x_0^2 - Ax_0y_0 + By_0^2 = \varepsilon, \tag{2}$$

where $\varepsilon = 1$ or $\varepsilon = -1$. If $x_0, y_0 > 0$, we may assume $x_0 \geqq y_0$. Indeed, for $B=1$ condition (1) is symmetric in x and y; and for $B=-1$, (2) gives

$$x_0^2 - y_0^2 \geqq A\,x_0y_0 - 1 \geqq 0$$

(because $x_0y_0 \neq 0$).

Furthermore, if $x_0 \geqq y_0 > 0$, then

$$x_0 = \frac{1}{2}\left(Ay_0 + \sqrt{A^2y_0^2 - 4By_0^2 + 4\varepsilon}\right).$$

For if

$$x_0 = \frac{1}{2}\left(Ay_0 - \sqrt{A^2y_0^2 - 4By_0^2 + 4\varepsilon}\right)$$

and $y_0 > 0$, then in case $B=-1$, $A>0$ we would get $x_0 \leqq 0$; and in case $B=1$, $A>3$ the condition $x_0 \geqq y_0$ would imply the inequality

$$Ay_0 - 2y_0 \geqq \sqrt{A^2y_0^2 - 4y_0^2 + 4\varepsilon},$$

equivalent to the inequality

$$(2-A)y_0^2 \geqq \varepsilon,$$

which is impossible for $A>3$, $y_0>0$ and $\varepsilon = \pm 1$.

It follows from (2) that the integers $x_1 = y_0$, $y_1 = ABy_0 - Bx_0$ also satisfy equation (1) since $B^2 = 1$ gives

$$x_1^2 - Ax_1y_1 + By_1^2 = y_0^2 - Ay_0(ABy_0 - Bx_0) + B(ABy_0 - Bx_0)^2$$
$$= Bx_0^2 - ABx_0y_0 + y_0^2 = B(x_0^2 - Ax_0y_0 + By_0^2) = B\varepsilon .$$

But if $x_0 > 0$ and $y_0 > 0$, then $x_1 > 0$ and

$$y_1 = ABy_0 - Bx_0 = ABy_0 - \frac{B}{2}(Ay_0 + \sqrt{A^2y_0^2 - 4By_0^2 + 4\varepsilon})$$

$$= \frac{B}{2}(Ay_0 - \sqrt{A^2y_0^2 - 4By_0^2 + 4\varepsilon} \geqq 0 \qquad (3)$$

($y_1 = 0$ only if $y_0 = 1$).
We show next that $y_1 < y_0$, except perhaps when $A = -B = 1$, $y_0 = 1$. In case $B = 1$, $A > 3$, using the form of y_1 given in (3), we get

$$y_1 = \frac{1}{2}(Ay_0 - \sqrt{A^2y_0^2 - 4y_0^2 + 4\varepsilon}) < \frac{1}{2}(Ay_0 - \sqrt{A^2y_0^2 - 4Ay_0^2 + 4y_0^2}) = y_0$$

since $A^2y_0^2 - 4y_0^2 + 4\varepsilon > A^2y_0^2 - 4Ay_0^2 + 4y_0^2$, i.e. $(A-2)y_0^2 > -\varepsilon$, clearly holds for $y_0 > 0$. And if $B = -1$, then

$$y_1 = \frac{1}{2}(\sqrt{A^2y_0^2 + 4y_0^2 + 4\varepsilon} - Ay_0) < \frac{1}{2}((A+2)y_0 - Ay_0) = y_0$$

since $4\varepsilon < 4Ay_0^2$, except perhaps if $A = 1, y_0 = 1$.
Continuing this procedure we construct the strictly decreasing sequences $y_0, y_1, y_2, \ldots$ and $x_0, x_1, x_2, \ldots$, where

$$x_i = y_{i-1} \quad \text{and} \quad y_i = ABy_{i-1} - Bx_{i-1} \quad \text{for} \quad i > 0 \qquad (4)$$

and $x_i > y_i \geqq 0$, if $y_{i-1} > 0$ (except perhaps if $y_{i-1} = 1$ in case $A = -B = 1$). Furthermore equality (1) holds for $x = x_i$, $y = y_i$.
The construction comes to an end when an index j is reached such that $y_j = 0$ (or $y_j = 1$ in case $A = -B = 1$). If $y_j = 0$, then $x_j = 1$, so that $y_j = R_0$ and $x_j = R_1$. But by (4) we can show that if $y_i = R_k$ and $x_i = R_{k+1}$ for some indices i and k, then $y_{i-1} = R_{k+1}$ and $x_{i-1} = Ay_{i-1} - By_i = AR_{k+1} - BR_k = R_{k+2}$ (since $B^2 = 1$); this shows that y_0, x_0 are also consecutive terms of sequence R. If $A = -B = 1$ and $y_j = 1$ for some index j, then $x_j = 2$, 1 or 0. But $(y_j, x_j) = (1,2) = (R_2, R_3)$, $(y_j, x_j) = (1,1) = (R_1, R_2)$ and $y_j = 1$, $x_j = 0$ imply that $y_{j-1} = 0 = R_0$, $x_{j-1} = 1 = R_1$; therefore we get as above that y_0, x_0 are also consecutive terms of sequence R.
This completes the proof of theorem 1.

Proof of theorem 2: Because of the conditions imposed on A and B, we have

$$x^2 - Axy + By^2 = 0$$

for integers x and y if and only if $x=y=0$. Therefore by theorem 1 for non-negative integers x, y, we have $f(x,y)=0$ if and only if $y=0$, $f(x,y)=y>0$ if and only if x and $y>0$ are two consecutive terms of the sequence R, and $f(x,y)<0$ in any other cases.

Remark: One can easily see that theorem 1 is valid for cases $A=1$, $B=1$ and $A=2$, $B=1$, but sequence R is degenerate in these cases. In case $A=3$, $B=1$ theorem 1 is false since $x=2$, $y=1$ is a solution of equation (1) and 2 is not a term of sequence R.

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REFERENCES

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Kleine Mitteilungen

Eine merkwürdige Familie von beweglichen Stabwerken

1. Sei $ABA'B'$ ein *gelenkiges Antiparallelogramm* mit den Seitenlängen $AB=A'B'=a$ und $AB'=A'B=d>a$. Wird es in seiner Ebene so bewegt, dass der Schnittpunkt O der Langseiten und die Symmetrieachse z festbleiben (Fig. 1), dann rollt bekanntlich eine Ellipse e mit den Brennpunkten A,B und der Hauptachse d auf einer kongruenten Ellipse e' mit den Brennpunkten A',B' gleitungslos ab, wie die Betrachtung des gemeinsamen Linienelements (O,z) lehrt; diese Tatsache bildet die kinematische Grundlage für elliptische Zahnräder [2]. Alle vier Gelenke des Antiparallelogramms wandern dabei auf einer gemeinsamen, aus zwei kongruenten Ovalen bestehenden Bahnkurve 6. Ordnung, wie in [5] gezeigt wurde.
Bezeichnet $r=OA$ den Radiusvektor des Punktes A und ψ den Richtungswinkel, gemessen von der zur z-Achse normalen x-Achse aus, so hat A die kartesischen Koordinaten $x=r\cos\psi$, $z=r\sin\psi$ und B die Koordinaten $\bar{x}=(d-r)\cos\psi$, $\bar{z}=(r-d)\sin\psi$. Die auf $AB=a$ bezügliche Distanzformel liefert dann für die *Bahnsextik* k die Polargleichung

$$r(d-r)\cos^2\psi=m^2 \quad \text{mit} \quad 4m^2=d^2-a^2, \tag{1}$$

welche auf die kartesische Gleichung

$$(x^2+z^2)(x^2+m^2)^2=d^2x^4 \tag{2}$$