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# Kleine Mitteilungen

## Remarks on the differences between consecutive primes

In problem 654, Journal of Recreational Mathematics, Harry Nelson asks: "What is the most likely difference between consecutive primes?" Here a difference is 'most likely' for primes $\leq n$ if it occurs at least as often as any other difference. For a discussion see J. Rec. Math. *11*, 231.

We first show that a well-known conjecture of Hardy and Littlewood implies that the most likely difference tends to infinity with $n$, so that there is no most probable difference independent of $n$.

In Hardy and Littlewood: On the expression of a number as a sum of primes. Collected Papers of G.H. Hardy, VI, p. 682; they conjecture that the number of solutions of

$$p_i - p_j = 2k, \qquad p_i \leq n, \tag{1}$$

equals

$$(c + o(1))\,\frac{n}{\log^2 n} \prod_{\substack{p \mid k \\ p\,\mathrm{odd}}} \frac{p-1}{p-2} \tag{2}$$

where $c$ is an absolute constant.
Let $p_i$ denote the $i$th prime; then (2) implies that the number of solutions of

$$p_{i+1} - p_i = 2k, \qquad p_{i+1} \leq n \tag{3}$$

also is of the form (2). Since every solution of (3) is a solution of (1) it is clear that the number cannot be greater than (2). On the other hand if we have a solution $p_i - p_j = 2k$ of (1) which is not a solution of (3), that is $i > j+1$, then we get a triple of primes

$$p_j, \qquad p_j + 2u, \qquad p_j + 2k; \qquad 1 \leq u < k. \tag{4}$$

From Brun's method it follows that the number of such triples with $p_j < n$ is less than

$$c_k n \prod_{k < p < n^{\varepsilon}} \left(1 - \frac{3}{p}\right) < c'_k \frac{n}{\log^3 n} \qquad (\varepsilon > 0) \tag{5}$$

for each fixed $u$, and hence $\leq c''_k n/\log^3 n$ for all triples in (4). Inequality (5) follows from the fact that the primes satisfying (4) exclude three residue classes $(\mathrm{mod}\, p)$ for $p > k$. Since the bound (5) is small compared to the estimate (2) it follows that (2) is also an estimate for the number of solutions in (3).
Now (2) implies that the most likely difference between consecutive primes goes to infinity with $n$. Denote the number of solutions of (3) by $f(n,k)$ and let $k_n$ be the minimum value of $k$ for which $f(n,k)$ is maximal.
Brun's method gives the well-known relation

$$f(n,k) < c_k n/\log^2 n \tag{6}$$

In view of the divergence of $\prod (p-1)/(p-2)$ estimates (2) and (6) imply that

$$f(n,k_n) \Big/ \frac{n}{\log^2 n} \to \infty \tag{7}$$

and (6), (7) imply $k_n \to \infty$ with $n$.

Of course the prime number theorem implies

$$f(n,k_n)\Big/\frac{n}{\log^2 n}>c>0 \tag{8}$$

for some fixed constant $c$, but this is not sufficient to prove that $k_n\to\infty$.
Next we ask: How fast does $k_n$ go to infinity? We conjecture that

$$k_n/(\log n)^{1-\varepsilon}\to\infty \quad \text{for every} \quad \varepsilon>0, \tag{9}$$

but

$$k_n/\log n\to 0. \tag{10}$$

Conjecture (2) is not strong enough to deduce (9) or (10). Perhaps they can be deduced from stronger plausible conjectures.
Let $l_n$ be the largest integer for which

$$f(n,k_n)=f(n,l_n) \tag{11}$$

then we still expect that

$$l_n/\log n\to 0. \tag{12}$$

Finally we conjecture that

$$f(n,k_n)\log^2 n/(n\log\log n)\to c>0. \tag{13}$$

Without unproven conjectures we cannot even improve (8) to

$$f(n,k_n)\log^2 n/n\to\infty. \qquad (14$$

Let $F(n,k)$ denote the number of solutions of

$$p_i-p_j=2k, \qquad p_i\le n. \tag{15}$$

Let $K_n$ be the least integer $k$ for which $F(n,k)$ is maximal.
Then

$$F(n,K_n)>c\,\frac{n\log\log n}{\log^2 n}. \tag{16}$$

In order to prove (16) choose $A=p_1\cdots p_m\le\sqrt{n}$. The primes $p$ with $p_m<p\le n$ are divided into $\varphi(A)$ residue classes $(\operatorname{mod} A)$ with $\lambda_s n/\log n$ in each class $s=1,\dots,\varphi(A)$. Here $\lambda_1+\cdots+\lambda_{\varphi(A)}=1+o(1)$. Thus the differences $p_i-p_j$ where $p_i,p_j$ belong to the same residue class $(\operatorname{mod} A)$ number

$$\left(\lambda_1^2+\cdots+\lambda_{\varphi(A)}^2+o(1)\right)\frac{n^2}{2\log^2 n}\geq\left(\frac{1}{\varphi(A)}+o(1)\right)\frac{n^2}{2\log^2 n}. \tag{17}$$

Since the number of integers $\leq n$ which are divisible by $A$ is $\leq n/A$ it follows from (17) that one of these integers has at least

$$\left(1+o(1)\right)\frac{A}{\varphi(A)}\cdot\frac{n}{2\log^2 n} \tag{18}$$

representations (15). Now

$$\frac{A}{\varphi(A)}=\prod_{i<(1/2)\log n}\left(1+\frac{1}{p_i-1}\right)>c\log\log n. \tag{19}$$

By a more careful application of this method we can prove that for any monotonically increasing $f(n)$, with $f(n)\to\infty$ as slowly as we please, there is a sequence $l_n<f(n)\log n$ for which

$$F(n,l_n)\log^2 n/n\to\infty. \tag{20}$$

We cannot prove (20) if we only assume $l_n<c\log n$ for some fixed $c$.

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## Eine Bemerkung zum gleichseitigen hyperbolischen Paraboloid als Bewegfläche bei Zylinderschrotungen

1. Die kinematische Beschreibung der i.a. $\infty^2$ vorhandenen (euklidischen) Zwangläufe, vermöge deren sich eine Quadrik als Bewegfläche eines Kegelschnitts erzeugen lässt, ist für den allgemeinen Fall offen [1]. Für das gleichseitige hyperbolische Paraboloid $\Phi$ gab Brauner [2] eine Zylinderschrotung an, bei der $\Phi$ durch eine gleichseitige Hyperbel erzeugt wird. Dies ist bereits die einzige (euklidische) Zylinderschrotung, bei der sich $\Phi$ als Bewegfläche eines Kegelschnitts erzeugen lässt [5].

Natürlich lassen sich solche Fragen auch für andere Transformationsscharen behandeln. In der Projektivkinematik kann dieser Problemkreis weitgehendst als gelöst gelten [4].

Ein neuer Gesichtspunkt ergibt sich durch Heranziehen nicht euklidischer Zwangläufe, wie etwa bei der Braunerschen Erzeugung der windschiefen (euklidischen) Böschungsflächen als $c$-Kanalflächen [3].

2. In diesem Sinne sollen für $\Phi$ zwei pseudoeuklidische Erzeugungsweisen angegeben werden. Wir zeigen: *Das gleichseitige hyperbolische Paraboloid lässt sich als pseudoeuklidische Bewegfläche bei Zylinderschrotungen erzeugen. Der erzeugende Kegelschnitt ist entweder eine Parabel oder eine gleichseitige Hyperbel.*

Der Beweis ergibt sich durch «Übertragung» der entsprechenden euklidischen Resultate, was für die Erzeugung durch eine *Parabel* insofern nicht verwunderlich ist, als $\Phi$ ein pseudoeuklidisches Drehparaboloid ist.
Das gleichseitige hyperbolische Paraboloid $\Phi$

$$a(\bar{x}^2-\bar{y}^2)=\bar{z} \tag{1}$$

wird durch die pseudoeuklidischen Zylinderschrotungen

$$\bar{x}=x\cosh t+(y+\mu)\sinh t, \qquad \bar{y}=x\sinh t+(y+\mu)\cosh t, \qquad \bar{z}=-a\mu^2+z \tag{2}$$

als Bewegfläche der Parabel

$$z=ax^2, \qquad y=0$$

erzeugt. Dabei ist $\mu=\mu(t)$ eine willkürliche, nicht konstante Parameterfunktion.
Auch die Braunersche Erzeugung [2] lässt sich übertragen. Offenbar wird die Fläche (1) durch die pseudoeuklidische Zylinderschrotung

$$\begin{aligned}
\bar{x}&=-\frac{1}{2a\sqrt{2}}\operatorname{tgh}t+\frac{x-y}{2}-\frac{x+y}{2}\cosh t+\frac{z}{\sqrt{2}}\sinh t\\
\bar{y}&=-\frac{1}{2a\sqrt{2}}\operatorname{tgh}t+\frac{x-y}{2}+\frac{x+y}{2}\cosh t-\frac{z}{\sqrt{2}}\sinh t\\
\bar{z}&=-ap^2\cosh t+\frac{x+y}{\sqrt{2}}\sinh t-z\cosh t
\end{aligned} \tag{3}$$

als Bewegfläche der gleichseitigen Hyperbel

$$x^2-y^2=p^2, \qquad z=0$$

erzeugt.

3. Es bereitet keine prinzipiellen Schwierigkeiten, die Resultate hinsichtlich der Bahnkurven, linearen Hüllgebilde usw. aus [2, 5] zu übertragen. Wir erwähnen: $\alpha$) Die Achsenzylinder der Schrotung (2) sind genau dann *kongruent,* wenn die pseudoeuklidische *Grundrissbewegung eine symmetrische Parabelrollung* ist. Die Darstellung kann so gewählt werden, dass

$$\mu=a\operatorname{tgh}\frac{t}{2} \quad \text{mit} \quad 0\neq a\in\mathbf{R}$$

gilt. $\beta$) Die Zylinderschrotung (2) hat genau dann pseudoeuklidische *Drehzylinder* als Achsenflächen, wenn die *Grundrissbewegung invers zur Hyperbelbewegung* – dem pseudoeuklidischen Analogon der Ellipsenbewegung – ist. $\gamma$) *Die Grundrissbewegung der Zylinderschrotung (3) ist invers zur Hyperbelbewegung.*

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## On the commutativity of two projections

The purpose of this note is the proof of the

**Proposition.** *For any two projections $P$ and $Q$ in a Hilbert space $H$, the commutativity relation $PQ=QP$ is equivalent to $PQP=QPQ$.*

Before we enter into the proof, let us make a few remarks. Obviously, the first equality implies the second. The reverse implication means that for $PQ=QP$ it is sufficient that $PQ$ has the same value for $Px$ as $QP$ has for $Qx$, for any $x \in H$. In other words, it permits an implication from the equality of positive, self-adjoint operators $PQP$ and $QPQ$ to prima facie more general operators $PQ$ and $QP$. Putting $A=PQ$, $A^*=QP$, the proposition may be restated as: $A=A^*$ is equivalent to $AA^*=A^*A$, i.e. for $A=PQ$ self-adjointness and normality are the same.
Finally, we remark that for one-dimensional projections $P$ and $Q$, $PQP=QPQ$ even implies $P=Q$ (unless $PQ=0$).
We need the following

**Lemma.** *Let $A$ and $B$ be bounded linear operators in $H$ such that*

$$(1)\ AB=BA, \quad (2)\ A^2=B^2 \quad \text{and} \quad (3)\ (A-B)=-(A-B)^*.$$

Then (4) $E$ commutes with any transformation that commutes with $A-B$, and (5) $A=(2E-I)B$, where $E$ is the orthogonal projection onto the null-space $M=N(A-B)$ of $A-B$.

Proof of the lemma (cf. [1], p. 424, theorem 23.3; note that in our proof $A$ and $B$ need not be self-adjoint!): Suppose that $C$ commutes with $A-B$. This implies $C(M) \subset M$.
From

$$C(A-B)=(A-B)C \Rightarrow (A-B)^*C^*=C^*(A-B)^*$$

and (3), we have

$$(A-B)C^* = C^*(A-B),$$

which implies $C^*(M) \subset M$. Therefore $C$ reduces $M$, i.e. $CE = EC$, proving (4).
From (1) and (2) we have

$$(A-B)(A+B) = A^2 - B^2 = 0,$$

i.e. (6) $E(A+B) = A+B$.
For any vector $z \in H$ write $z = x+y$, where $x \in M$ and $y \in M^\perp$. It follows

$$E(A-B)z = E(A-B)x + E(A-B)y.$$

The first term on the right is zero, because $x \in M = N(A-B)$ and the second is zero, because $E$ commutes with $A-B$, according to (4).
Hence

$$(7)\quad E(A-B) = 0.$$

Combining (6) and (7), gives $E(A+B) - E(A-B) = A+B$ or $A = 2EB - B = (2E-I)B$, which proves (5).
We wish to apply the lemma for $A = PQ$, $B = QP$.
Assumption (1) is the same as $PQP = QPQ$: Using this, and observing

$$(8)\quad (PQ)^2 = PQPQ = PPQP = PQP = QPQ = (QP)^2,$$

we note that (2) is fulfilled.
Moreover, $(PQ - QP)^* = QP - PQ = -(PQ - QP)$, which is assumption (3) of our lemma.
Proof of the proposition: From $(PQP)^2 = PQPPQP = PQPQP = PQQPQ = (PQ)^2$ and (8), we see that $PQP = QPQ$ must be a projection. Writing $M_P$ and $M_Q$ for the ranges of $P$ and $Q$, and denoting by $\vee$ and $\wedge$ the closed span and the intersection of subspaces in $H$, we claim that $PQP$ is the projection onto $M_P \wedge M_Q$.
This may be seen from

$$PQP = (PQP)^2 = (PQ)^2, \qquad \text{i.e.} \quad PQP = (PQ)^{2k}, \qquad k \geq 1,$$

and from the fact that the projection onto $M_P \wedge M_Q$ is given by the limit of $(PQ)^n$, $n \to \infty$.
A more direct proof proceeds as follows:
a) for $z \in M_P \wedge M_Q$ obviously $PQPz = z$;
b) for $z \in (M_P \wedge M_Q)^\perp = M_P^\perp \vee M_Q^\perp$ write $z = \lim_n (x_n + y_n)$, where $x_n \in M_P^\perp$ and $y_n \in M_Q^\perp$, so that $PQPz = \lim_n (PQPx_n + PQPy_n) = \lim_n (0 + QPQy_n) = 0$, using continuity of $PQP$.
We prove now $PQ = QP$.

If $z \in M_P \wedge M_Q$, trivially $PQPz = QPz = z - z = 0$.
If $z \in M_P^{\perp} \vee M_Q^{\perp}$, write $z = \lim(x_n + y_n)$, as above.
Using (5) of our lemma gives

$$PQ = (2E - I)QP \quad \text{and} \quad QP = (2E - I)PQ$$

where $E$ is the orthogonal projection onto

$$N(PQ - QP) = N(QP - PQ)$$

(note that the assumptions of the lemma are symmetrical in $A, B$).
Therefore, using continuity,

$$PQz = \lim_n (PQx_n + PQy_n) = \lim_n PQx_n = \lim_n (2E - I)QPx_n = 0,$$

similarly $QPz = 0$. Hence, $PQ$ and $QP$ coincide on $H$.

Remarks:
a) We note that the above proof of $PQ = QP$ did not explicitly make use of the fact that $PQP$ is the projection onto $M_P \wedge M_Q$ (see first half of the proof); indeed, this follows immediately from $PQ = QP$. We have given a slightly redundant proof in order to show where the lemma enters the argument.

b) It should be noted that the proposition is a special case of a rather deep theorem by Fuglede-Putnam-Rosenblum (cf. [3], p. 300, theorem 12.16, for Rosenblum's proof):
Assume that $A, B, T$ are bounded transformations on $H$, $A$ and $B$ are normal, and $AT = TB$. Then $A^*T = TB^*$.
Taking $A = PQ$, $B = QP$, $T = P$ yields our proposition.

c) This note was suggested by Mittelstaedt's work on quantum probability theory ([2], see esp. p. 210-218), where the probability of a yes-no-event represented by $P$ occurring after the event given by $Q$ is defined by $w_\varphi(P,Q) = \langle \varphi, PQP\varphi \rangle$, $\varphi$ denoting the state of the physical system. The proposition says that if $w_\varphi(P,Q) = w_\varphi(Q,P)$ for all states, then there can be no interference, i.e. $P$ and $Q$ commute.

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