The number of triangles in a tringulation of a set of points in the plane

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Benützung der zu $\tilde{v}_1$ parallelen Elationsachse durch P kann der Parabelscheitel nach Abschnitt 3 konstruiert werden.

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The number of triangles in a triangulation of a set of points in the plane[1])

1. Introduction

Our terminology and notation will be standard except as indicated. A good reference for undefined graph theoretic terms is [3].

In [1, 2] the authors discussed the question of the number of 3-cycles which could be present in a planar graph on p points. In this paper, we want to consider essentially the same question when the p points are in *fixed positions* in the plane. We will show that this restriction does not limit the possible range of the number of 3-cycles unless the p points are arranged in a unique, easily characterized configuration.

2. Statement of the problem and main results

Begin with a set P of $p \geq 5$ points in the plane, with no three of the points collinear. Suppose we draw straight line segments between pairs of points in P subject only to the restriction that these segments do not intersect except at the points of P themselves, until it is impossible to add more segments in this manner. We call this collection of line segments a *triangulation* of P (since all the finite regions into which these segments divide the plane are triangles). We will generally use T to denote a triangulation of P. Note in particular that the line segments comprising the boundary of the convex hull of P will be included in every triangulation T of P.

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We want to consider the number of triangles (or 3-cycles) in various triangulations of P. To this end, let h denote the number of extreme points of the convex hull of P. (For brevity in the sequel, we will term this collection of h points the *extreme points* of P.) In any triangulation T of P, it follows by Euler's well-known formula that the number of 3-cycles each of which bounds a region (i.e., contains points of P in either its interior or exterior, but not both) will be precisely $2p$-h-2 if $h>3$, and $2p$-4 if $h=3$. In addition, however, a triangulation T of P may have 3-cycles containing points of P in both their interior and exterior. We will call such 3-cycles *separating*. In [1, 2], it was shown that the number of separating 3-cycles must be between 0 and either $p-h$ if $h>3$ or $p-4$ if $h=3$. Our goal is to show that except for the two cases described in the statement of the theorem below, it is always possible to triangulate P so as to obtain any number of separating 3-cycles in the indicated range.
We begin with the following result.

Lemma. *Let P be a set of $p \geq 5$ points in the plane, with no three of the points collinear. Suppose P has h extreme points. Then there is a triangulation T of P without separating triangles, unless $h=3$ and P has an extreme point x such that $P-x$ has $p-1$ extreme points (see fig. 1). In this exceptional case, any triangulation of P contains exactly $p-4$ separating triangles.*
Moreover, if $h<p$ and it is possible to triangulate P without separating triangles, then it is possible to obtain such a triangulation with no 'chords' between extreme points of P (i.e., with no line segments between extreme points of P except those comprising the boundary of the convex hull of P).

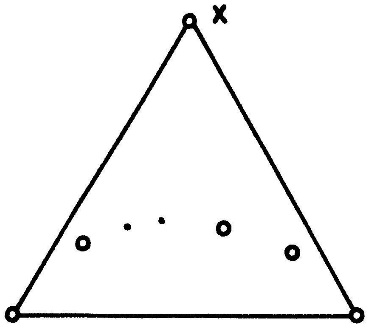

Figure 1

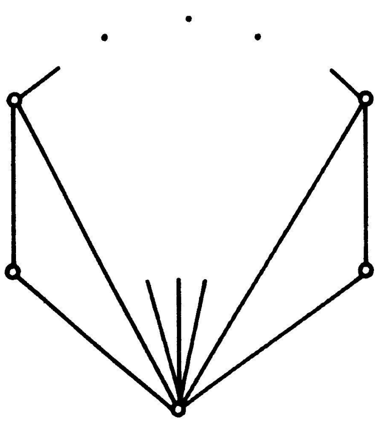
Figure 2

Proof: Observe first that if $h=p$, the desired triangulation is trivial (see fig. 2). Hence we assume $h<p$ in the rest of the proof.
Noting that the lemma is readily verified for $p=5$, we proceed by induction on p. Suppose first that $h\geq 4$. It is then easy to see that we can choose an extreme point x of P such that $P-x$ contains say $h'\geq 4$ extreme points. If $h'<p-1$, then by our induction hypothesis, we can triangulate $P-x$ without separating triangles or chords between extreme points of $P-x$. It is then a simple matter to obtain the desired triangulation of P (see fig. 3). If $h'=p-1$, then $P-x$ has extreme points y,z positioned as shown in figure 4. (If y (resp., z) did not exist, we would have $h=p$ (resp., $h=3$), contrary to what we have assumed.) It is then a simple matter to complete the desired triangulation of P (see fig. 4).

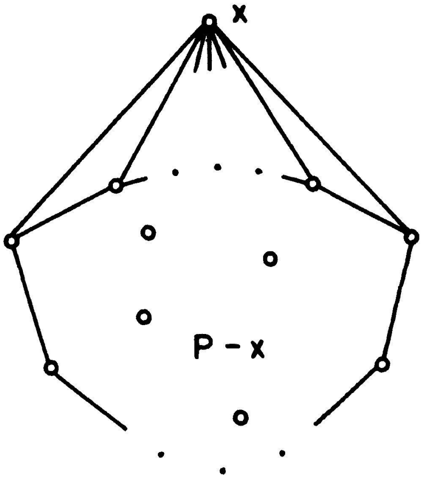

Figure 3

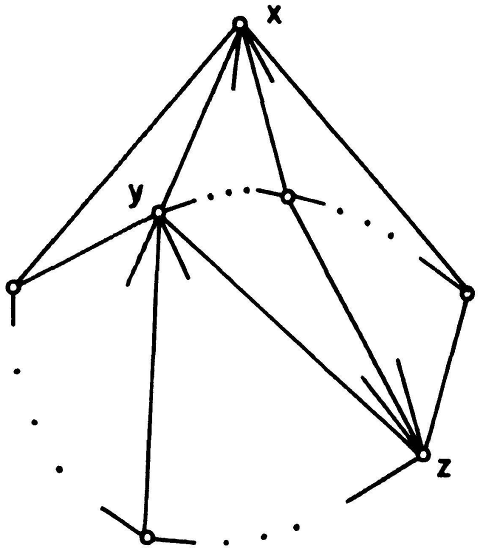

Figure 4

Suppose, therefore, that $h=3$. Let x_1, x_2, x_3 be the extreme points of P. If $P-x_i$ has h_i extreme points, where $4 \le h_i < p-1$, for some i, then we can triangulate P without separating triangles as in the last paragraph. Otherwise, $P-x_i$ has three extreme points for each i. In particular, let x_2, x_3 and say x_4 be the extreme points of $P-x_1$. Suppose first that $P-x_1-x_i$ does not have $p-2$ extreme points, for $i=2,3,4$. Then by the induction hypothesis, we can triangulate $P-x_1$ without separating 3-cycles. Call this triangulation T'. It is easy to see that there will be a point x in the interior of $x_2x_3x_4$ such that xx_ix_4 is a nonseparating 3-cycle of T', and $xx_ix_1x_4$ is convex, for either $i=2$ or 3; without loss of generality, suppose this occurs for $i=2$ (see fig. 5a). We obtain a triangulation T of P without separating 3-cycles from T' as follows: Remove the line segment x_2x_4 from T', and add the segments x_1x, x_1x_2, x_1x_3 and x_1x_4 (see fig. 5b). It is easily seen that the only way T could contain a separating 3-cycle is if the line segment xx_3 belonged to T. But in that case, either xx_2x_3 or xx_3x_4 would be a separating 3-cycle in T', a contradiction.

Suppose, therefore, that $P-x_1-x_i$ has $p-2$ extreme points, for some $i=2,3$ or 4. We need to consider essentially two cases. If $P-x_1-x_2$ has $p-2$ extreme points

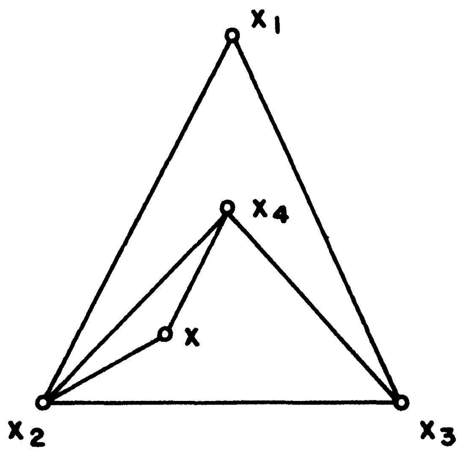

Figure 5a

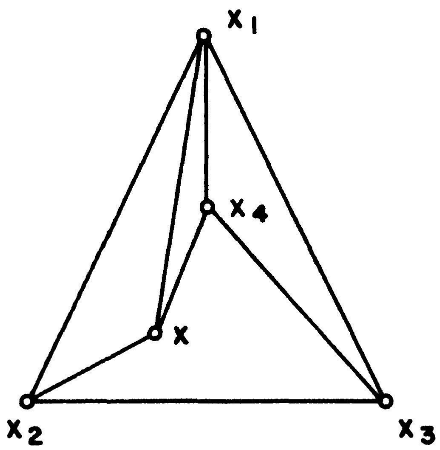

Figure 5b

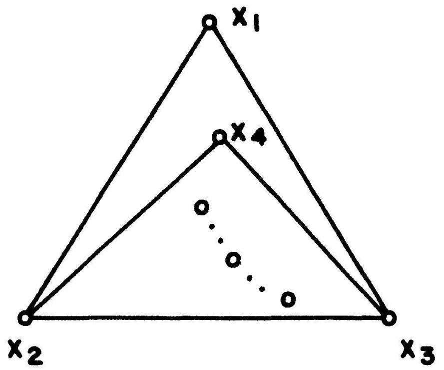

Figure 6a

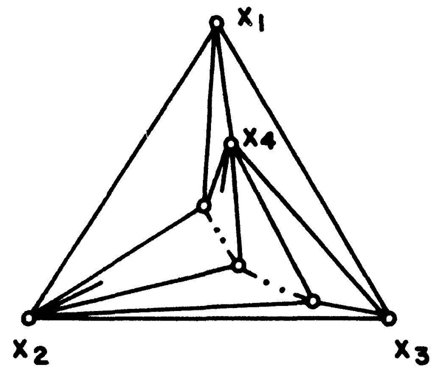

Figure 6b

(see fig. 6a), then we can triangulate P without separating 3-cycles as shown in figure 6b (assuming of course that $P-x_2$ does not have $p-1$ extreme points).

On the other hand, if $P-x_1-x_4$ has $p-2$ extreme points (see fig. 7a), we can triangulate P without separating triangles as shown in figure 7b. This completes the proof for all but the exceptional case.

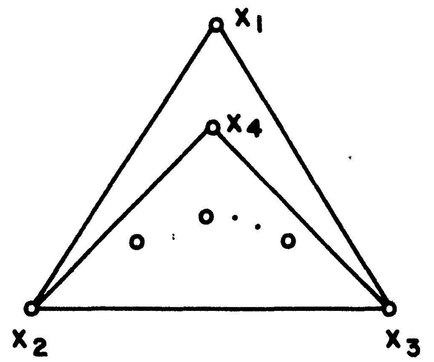

Figure 7a

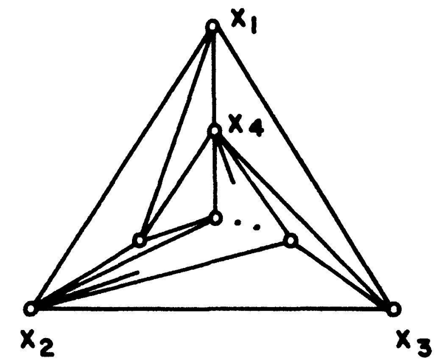

Figure 7b

Consider, therefore, the situation when $h=3$ and P has an extreme point x such that $P-x$ contains $p-1$ extreme points. Let y,z be the other extreme points of P. Then in any triangulation T of P, there will be a point w inside xyz such that wyz is a nonseparating 3-cycle in T (see fig. 8). It is then easy to see that the line segment wx must also belong to T.

Consider the sets of points P_1 and P_2 inside or on the 3-cycles wxy and wxz, respectively. It is easy to see that P_i-x has $|P_i|-1$ extreme points, for $i=1,2$. If

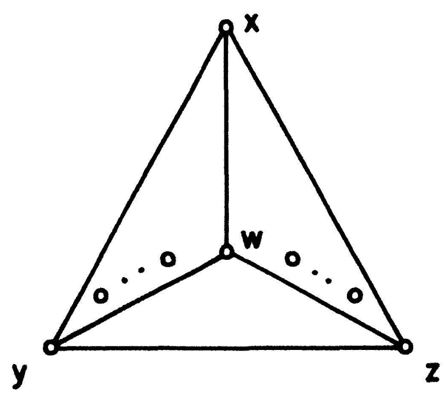

Figure 8

$|P_i| \geq 5$ for $i=1,2$, it follows by the induction hypothesis that any triangulation T_i of P_i must contain $|P_i|-4$ separating 3-cycles. Moreover, the only other separating 3-cycles in T (besides those in T_1 and T_2) would be wxy and wxz. Hence the number of separating 3-cycles in T will be

$$(|P_1|-4)+(|P_2|-4)+2=p-4,$$

as asserted. The cases when $|P_i|=3$ or 4 are similar, and are, therefore, omitted. This completes the proof of the lemma.

We can now state our main result.

Theorem. *Let P be a set of $p \geq 5$ points in the plane with no three of the points collinear. Suppose P has h extreme points. Let $\Delta(T)$ denote the number of 3-cycles in a triangulation T of P. Then*

1. *If $h \geq 4$,*

$$2p-h-2 \leq \Delta(T) \leq 3p-2h-2.$$

Moreover, if a is any number in the indicated range, there is a triangulation T of P with $\Delta(T)=a$.

2. *If $h=3$,*

$$2p-4 \leq \Delta(T) \leq 3p-8.$$

Moreover, if a is any number in the indicated range, then there is a triangulation T of P with $\Delta(T)=a$ unless either

a) *$a=3p-9$, or*

b) *P has an extreme point x such that $p-x$ has $p-1$ extreme points. (In this case, $\Delta(T)=3p-8$ for every triangulation T of P.)*

Proof: For case 2a, it was shown in [1] that a maximal planar graph on p points cannot contain exactly $3p-9$ 3-cycles. Moreover, case 2b was covered in the preceeding lemma. It only remains, therefore, to treat the nonexceptional cases.

Suppose first, therefore, that $h \geq 4$ as in case 1. Choose any $(3p-2h-2-a)$ non-extreme points of P, and let P' denote these points together with the h extreme points of P. Triangulate P' without separating 3-cycles (this is possible by the lemma). At this stage, we have exactly $(2(3p-h-2-a)-h-2)$ 3-cycles. Then recursively join each of the remaining $(a-2p+h+2)$ points of $P-P'$ by line segments to the three points of the triangle in which it occurs. The resulting triangulation of P contains $2(3p-h-2-a)-h-2+3(a-2p+h+2)=a$ 3-cycles as desired.

Thus suppose $h=3$ as in case 2, but $a \neq 3p-9$. If $a=3p-8$, recursively draw line segments between each nonextreme point of P (taken in any order) and the three points of the triangle in which it occurs. If $a \leq 3p-10$, let us suppose for the moment that we can choose a set S of three nonextreme points in P such that S, together with the extreme points of P, comprise a set of six points not of the type excluded by case 2b. Then choose arbitrarily an additional $3p-10-a$ nonextreme points of P. These additional points, together with S and the extreme points of P, will from a set P' of $3p-4-a$ points which again are not of the type excluded by case 2b. We can, therefore, triangulate P' without separating 3-cycles; at this stage we have exactly $(2(3p-4-a)-4)$ 3-cycles. Now recursively draw line segments between each of the remaining $(a-2p+4)$ points of $P-P'$ and the three points of the triangle in which it occurs to obtain a triangulation of P with exactly $2(3p-4-a)-4+3(a-2p+4)=a$ 3-cycles.

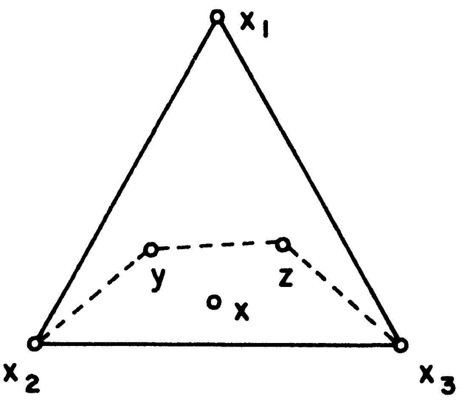

Figure 9

To complete the proof, we need to establish the existence of the set S. Let the extreme points of P be x_1, x_2, x_3. Suppose that $P-x_i$ has three extreme points, say y_i, x_j and x_k, for each i. Then we can take $S=\{y_1, y_2, y_3\}$. Otherwise, suppose that say $P-x_1$ has at least four extreme points. Since $P-x_1$ does not have $p-1$ extreme points, let x be a nonextreme point of $P-x_1$. Then it is easy to see there exist extreme points y, z of $P-x_1$ such that x is positioned in the interior of the convex quadrilateral $x_2 y z x_3$ (see fig. 9). We can then take $S=\{x, y, z\}$.
This completes the proof of the theorem.

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