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The impossibility of a tesselation of the plane into equilateral triangles whose sidelengths are mutually different, one of them being minimal

Theorem. *There is no tesselation of the euclidean plane* $\mathbf{R}^2$ *into equilateral triangles whose sidelengths are mutually different, one of them being minimal.*

Proof: Assume that there is such a tesselation of $\mathbf{R}^2$ into equilateral triangles $t_i, i \in I$, where I is an arbitrary set of indices. We shall eventually see that this assumption leads to a contradiction.

For $i \in I$, let l_i denote the sidelength of t_i. Let l be the minimum of the sidelengths. By scaling we can attain $l = 1$. Then the area of each triangle is at least

$$\frac{1}{4} \cdot \sqrt{3}\,.$$

Therefore the triangle can be enumerated and we can assume $I = \mathbf{N}$ and $l_1 = 1$.

For each $i \in \mathbf{N}$ led d_i denote the boundary of t_i. Further we define D to be the 'grid' of the tesselation: $D = \cup \{d_i, i \in \mathbf{N}\}$.

If n and ε are positive real numbers, the points

$$P_1 = \left(-\frac{n}{2} - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\sqrt{3}\right), \quad P_2 = \left(-\frac{n}{2}, 0\right), \quad P_3 = \left(\frac{n}{2} - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\sqrt{3}\right)$$

$$\text{and} \quad P_4 = \left(\frac{n}{2}, 0\right)$$

define a parallelogram in $\mathbf{R}^2$ with the sidelengths ε and n and with angles of 60 and 120 degrees. Let $S(n,\varepsilon)$ denote the union of the three sides $\overline{P_1P_2}$, $\overline{P_2P_3}$ and $\overline{P_3P_4}$ (fig. 1).

More generally we denote by $S(n,\varepsilon)$ any subset of the plane that can be obtained from this special $S(n,\varepsilon)$ by translations, rotations and reflections. If any such $S(n,\varepsilon)$ is given, let R_n denote the interior of the associated parallelogram.

We show that there are a strongly decreasing sequence of positive numbers $n_j, j \in \mathbf{N}$, with $n_j \leq n_{j-1} - 1\ (j \geq 2)$ and for each j a number $\varepsilon_j \in (0, \infty)$ as well as a set $S_{n_j} = S(n_j, \varepsilon_j)$, contained in D, such that S_{n_j} suffices one of the following two properties (or both):

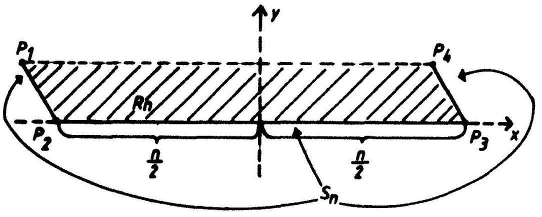

Figure 1

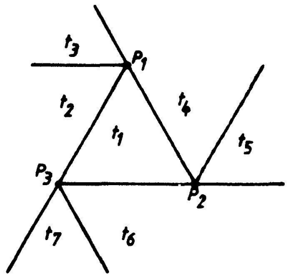

Figure 2

a) There is a natural number k depending on j such that $n_j = l_k$ and $R_{l_k} \cap t_k = \phi$.
b) The 'base' of S_{n_j} $(\overline{P_2 P_3}$ in fig. 1) is not a side of a triangle of the tesselation.

The numbers n_j and the associated sets S_{n_j} will be constructed by induction on j.
Since $n_j \leq n_{j-1} - 1$ for $j \geq 2$, some of the numbers n_j must be negative. But we assumed that each n_j is positive. Thus we get a contradiction and the theorem is proved.
The induction is performed in two steps.

First step

We show that there is a subset S_{n_1} of D for some positive real number $n = n_1$ which has property a) or property b).
Look at figure 2. Let P_1, P_2 and P_3 be the vertices of t_1. Since t_1 is the smallest triangle of the tesselation, each neighbor of t_1 (i.e. each triangle that shares a boundary segment of positive measure with t_1) must be larger than t_1. It is obvious that this is only possible if there are only three neighbors of t_1, say t_2, t_4 and t_6, and if (up to symmetry) they are arranged like in figure 2.
Hence there exist three triangles in the tesselation that have only a vertex in common with t_1. We can assume that these are the triangle t_3, t_5 and t_7 and that the triangles t_2, t_3, t_4, t_5, t_6 and t_7 are arranged around t_1 in a clockwise manner.
Let us call t_i, $i = 2, \ldots, 7$, the *surrounding triangles*. The vertices of the surrounding triangles that are not points of t_1 may be called the *outer corners*. Clearly each surrounding triangle has exactly two outer corners. For each surrounding triangle t_k, let P_{2k} and P_{2k+1} be the outer corners written clockwise (see fig. 3). Since the triangles of the tesselation have pairwise different sizes, we get the inequalities $P_5 \neq P_6$,

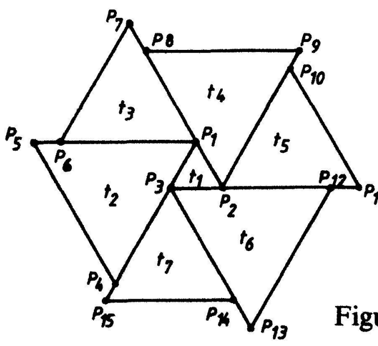

Figure 3

$P_9 \neq P_{10}$ and $P_{13} \neq P_{14}$.
We now have to distinguish the two cases wether t_2 is larger or smaller than t_3.

Case 1. We assume that $l_2 > l_3$.
Consider figure 3. If D would contain a horizontal straight line whose right endpoint is P_7, the points P_6 and P_7 would define a set $S_{l_3} \subset D$ with $R_{l_3} \cap t_3 = \phi$.
If this is not the case, D contains a straight line which elongates $\overline{P_6P_7}$ over P_7 and therefore a straight line whose left endpoint is P_7, too.
If $P_7 \neq P_8$, the points P_2 and P_7 define a set $S_{n_1} \subset D$ that has property b). So we may assume $P_7 = P_8$.
Now consider all systems of straight lines that may start at P_9 (fig. 4). In the cases i) and ii) of figure 4 there is a set $S_{l_4} \subset D$ with $R_{l_4} \cap t_4 = \phi$, defined by P_7 and P_9. In case iii) there is a set $S_{l_4} \subset D$ with $R_{l_4} \cap t_4 = \phi$, defined by $\overline{P_2P_{11}}, \overline{P_2P_9}$ and the elongation of P_8P_9 over P_9. Therefore we may assume that the lines starting from P_9 look as in figure 4 iv).
This implies $l_4 > l_5$.
Two further quite similar argumentations - we now have $l_4 > l_5$ like we had $l_2 > l_3$, and we will get $l_6 > l_7$ - show that if there is not a set $S_{n_1} \subset D$ with property a) or b), the conditions $P_{11} = P_{12}$ and $P_4 = P_{15}$ must be fulfilled. Moreover, the line systems starting at P_5, P_9 and P_{13} are of type iv) of figure 4 (see fig. 5).
Let d_1, d_2 and d_3 denote the (positive) lengths of $\overline{P_5P_6}, \overline{P_9P_{10}}$ and $\overline{P_{13}P_{14}}$, respectively. Then each d_i, $i = 1,2,3$ is a sum of sidelengths of some triangles of the tesselation. Therefore $d_i > l_1$ for $i = 1,2,3$. Hence $d_1 + d_2 + d_3 > 3\,l_1$.
Comparing the sidelength of the seven triangles t_1, t_2, t_3, t_4, t_5, t_6 and t_7 yields the equations $l_2 = l_1 + l_7$, $l_2 = l_3 + d_1$, $l_4 = l_1 + l_3$, $l_4 = l_5 + d_2$, $l_6 = l_1 + l_5$, $l_6 = l_7 + d_3$. They tell us that $3\,l_1 = d_1 + d_2 + d_3$ in contradiction to the last inequality.
Therefore in case 1 there must be a set $S_{n_1} \subset D$ with property a) or b).

Case 2. We assume that $l_2 < l_3$.
If there is not a set $S_{l_2} \subset D$ with $R_{l_2} \cap t_2 = \phi$ defined by P_4 and P_5, the situation at P_4 can only be as indicated in figure 6. Then the points P_1 and P_4 define a set $S_{l_2} \subset D$ with $P_{l_2} \cap t_2 = \phi$.

Second step

A set $S_n (= S_{n_j}) \subset D$ may be given that has property a) or b). It will be shown that there is a set $S_m \subset D$ with $m \leq n-1$ which has property a). We then put $n_{j+1} := m$.

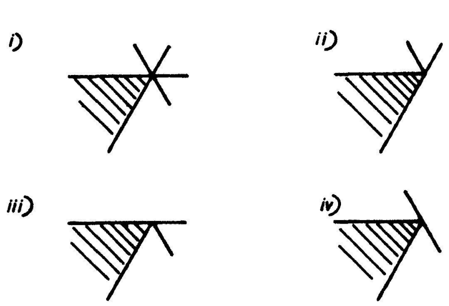

Figure 4

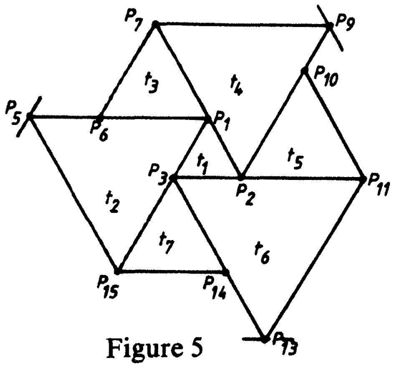

Figure 5

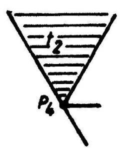

Figure 6

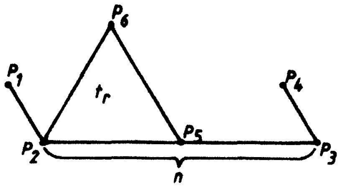

Figure 7a

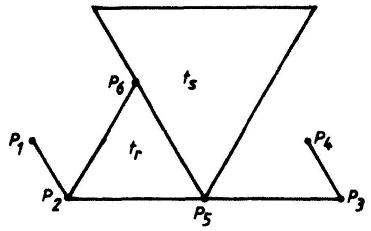

Figure 7b

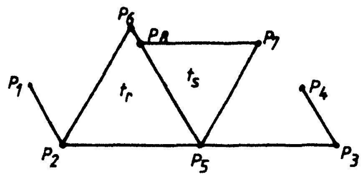

Figure 7c

After a suitable translation, rotation and reflection if necessary we can assume that the given S_n has the position shown in figure 1. With the notations of figure 1, let t_r be the triangle of the tesselation which 'stands on the base of S_n at the left corner', which means that t_r is defined by the three conditions $P_2 \in t_r$, $R_n \cap t_r \neq \phi$, $\overline{P_2 P_3}$ contains a side of t_r (see fig. 7a). The vertices of t_r may be P_2, P_5 and P_6, where $P_5 \in \overline{P_2 P_3}$. Since S_n has property a) or b), P_3 is different from P_5.
Let t_s denote the neighbor of t_r at the side $\overline{P_5 P_6}$ such that $P_5 \in t_s$. We now have to distinguish whether l_s is greater or less than l_r.

Case 1. If $l_s > l_r$, the points P_2 and P_6 define a set $S_{l_r} \subset D$ with $R_{l_r} \cap t_r = \phi$ (see fig. 7b). Moreover, $l_r \leq n-1$ because $S_n \subset D$ and $l_1 = 1$. So we can define $m = l_r$.

Case 2. If $l_s < l_r$, let P_7 denote the vertex of t_s that is not a point of t_r and let P_8 denote its third vertex (fig. 7c). An examination of all possible line systems starting at P_7 (they are listed in fig. 4) shows that in each case there is a set $S_{l_s} \subset D$ with $R_{l_s} \cap t_s = \phi$, either defined by the points P_5 and P_7 or by P_7 and P_8. Moreover, $l_s < l_r \leq n-1$. So we can define $m = l_s$.
Hence the induction step is completed, q.e.d.

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Quelques considérations concernant le problème de l'aiguille de Buffon dans l'espace euclidien E_n

0. Soit E_n l'espace euclidien à n dimensions de coordonnées $x_1, \ldots, x_n$.
La mesure élémentaire cinématique dans E_n, invariante par rapport au groupe de mouvements euclidiens, est [1]:

$$dK = dP \wedge dO_{n-1} \wedge \cdots \wedge dO_1, \tag{1}$$