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A basis of the set of sequences satisfying a given m-th order linear recurrence

Given fixed numbers $u_0, u_1, \ldots, u_{m-1}$, define u_{m+n} for every $n \geq 0$ by means of the m preceding terms with the rule

$$u_{m+n} - k_1 u_{m+n-1} - \ldots - k_m u_n = 0 \quad \text{with} \quad k_m \neq 0. \tag{1}$$

Here we are working within a fixed field F of characteristic 0. It is easy to see that with respect to addition and multiplication by scalars of F, the set $\mathscr{S}_m$ of sequences of elements of F satisfying (1) forms a vector space over F. The purpose of this note is to exhibit a basis for $\mathscr{S}_m$. The result can in fact be found in the literature, but the details of a proof involving linear algebra may be useful to the lazy persons (like myself) interested in seeing the details written down without working them out.
Let us first state a proposition which can be found in any classical reference on finite differences; see for instance [2], [6] or [5].

Proposition 1. *Suppose that* $\{u_{n1}\}_{n\in\mathbf{N}}, \{u_{n2}\}_{n\in\mathbf{N}}, \ldots, \{u_{nm}\}_{n\in\mathbf{N}}$ *are* m *sequences satisfying* (1). *Let* $a_1, a_2, \ldots, a_m$ *be given scalars. Then* $\{a_1 u_{n1} + a_2 u_{n2} + \ldots + a_m u_{nm}\}_{n\in\mathbf{N}}$ *is also a sequence satisfying* (1).
Conversely, if $\{u_{n1}\}_{n\in\mathbf{N}}, \{u_{n2}\}_{n\in\mathbf{N}}, \ldots, \{u_{nm}\}_{n\in\mathbf{N}}$ *are* m *sequences satisfying* (1) *and such that*

$$\det[u_{ij}] \neq 0 \quad (\textit{with } 0 \leq i \leq m-1,\ 1 \leq j \leq m),$$

then any sequence $\{u_n\}_{n\in\mathbf{N}}$ *satisfying* (1) *can be written as*

$$\{u_n\}_{n\in\mathbf{N}} = \{a_1 u_{n1} + a_2 u_{n2} + \ldots + a_m u_{nm}\}_{n\in\mathbf{N}}$$

for a unique choice of scalars $a_1, a_2, \ldots, a_m$.

Corollary 2. *Suppose that* $\{u_{n1}\}_{n\in\mathbf{N}}, \{u_{n2}\}_{n\in\mathbf{N}}, \ldots, \{u_{nm}\}_{n\in\mathbf{N}}$ *are* m *sequences satisfying* (1) *and such that for* $0 \leqq i \leqq m-1$ *and for* $1 \leqq j \leqq m$,

$$u_{ij} = \textit{Kronecker } \delta_{i+1,j} = \begin{cases} 1 & \textit{if } i+1=j, \\ 0 & \textit{elsewhere}. \end{cases}$$

Then they form a basis of $\mathscr{S}_m$.

Proof. We have $\det[u_{ij}] = \det I_m = 1$. Q.E.D.

Associated with the recursive sequence (1) is the following polynomial

$$f(X) = X^m - k_1 X^{m-1} - \ldots - k_{m-1} X - k_m,$$

whose roots are nonzero since $k_m \neq 0$.

Theorem 3. *Suppose that $f(x)$ is a polynomial of degree m,*

$$f(X)=\prod_{i=1}^{r}(X-\alpha_i)^{s_i}, \tag{2}$$

such that the roots α_i are all distinct with multiplicity s_i. Then the $s_1+s_2+\ldots+s_r=m$ sequences

$$\{\alpha_1^n\}_{n\in\mathbf{N}}, \left\{\frac{n}{1!}\alpha_1^{n-1}\right\}_{n\in\mathbf{N}}, \ldots, \left\{\frac{n(n-1)\ldots(n-s_1+2)}{(s_1-1)!}\alpha_1^{n-s_1+1}\right\}_{n\in\mathbf{N}},$$

$$\{\alpha_2^n\}_{n\in\mathbf{N}}, \left\{\frac{n}{1!}\alpha_2^{n-1}\right\}_{n\in\mathbf{N}}, \ldots, \left\{\frac{n(n-1)\ldots(n-s_2+2)}{(s_2-1)!}\alpha_2^{n-s_2+1}\right\}_{n\in\mathbf{N}},$$

$$\ldots$$

$$\{\alpha_r^n\}_{n\in\mathbf{N}}, \left\{\frac{n}{1!}\alpha_r^{n-1}\right\}_{n\in\mathbf{N}}, \ldots, \left\{\frac{n(n-1)\ldots(n-s_r+2)}{(s_r-1)!}\alpha_r^{n-s_r+1}\right\}_{n\in\mathbf{N}}$$

form a basis of $\mathscr{S}_m$.

Proof. Let it be clear that if $s_k=1$, then we take only $\{\alpha_k^n\}_{n\in\mathbf{N}}$.
To see that we have a basis of $\mathscr{S}_m$, let V be the $m\times m$ matrix which we obtain by writing as elements of its k-th row the first m elements of the above k-th sequence. If for $1\leqq k\leqq r$, we let $i_k=s_1+s_2+\ldots+s_{k-1}$ with $i_1=0$, then the (i,j) element of V is

$$v_{i_k+i,j}=\frac{1}{(i-1)!}\cdot\frac{d^{i-1}\alpha_k^{j-1}}{d\alpha_k^{i-1}} \quad \text{for } 1\leqq i\leqq s_k. \tag{3}$$

Here d^iX/dX^i is the i-th derivative. We want to prove that

$$|V|=\begin{cases} 1 & \text{for } r=1, \\ \prod_{r\geqq j>i\geqq 1}(\alpha_j-\alpha_i)^{s_js_i} & \text{elsewhere;} \end{cases} \tag{4}$$

we will then be finished since the non vanishing of $|V|$ implies that the above sequences form indeed a basis of $\mathscr{S}_m$.
Let us proceed by induction on m. Formula (4) is correct for $m=1,2$ as is easily seen and for any m when $r=1$. Suppose it true for $m-1$, i.e. for h distinct roots of multiplicity $t_1, t_2, \ldots, t_h$ respectively such that $t_1+\ldots+t_h=m-1$.
Make on V the following elementary row operations: for $i=m, m-1, \ldots, 2$ (in this order), subtract from the i-th row α_1 times the $(i-1)$-th row. Delete from the obtained matrix the first row and the first column and call V^* the resulting $(m-1)\times(m-1)$ matrix.
For $j=s_1, s_1+s_2, \ldots, s_1+s_2+\ldots+s_{r-1}$, we can factorize from the j-th column of V^* the factors $(\alpha_2-\alpha_1), (\alpha_3-\alpha_1), \ldots, (\alpha_r-\alpha_1)$ respectively to obtain a matrix M verifying

$$|V|=|V^*|=\left(\prod_{j=2}^{r}(\alpha_j-\alpha_1)\right)|M|.$$

For any $k \geqq 2$ such that $s_k \geqq 2$, make on M the following elementary column operations: for $j = s_1 + \ldots + s_{k-1} + 1,\ s_1 + \ldots + s_{k-1} + 2, \ldots, s_1 + \ldots + s_{k-1} + s_k - 1$ (in this order), subtract from the i-th column the $(j-1)$-th column and factorize $(\alpha_k - \alpha_1)$. We then obtain a matrix W such that

$$|V| = \left(\prod_{j=2}^{r} (\alpha_j - \alpha_1)^{s_j}\right) |W|, \tag{5}$$

where W is the very matrix we get by deleting the last row and the s_1-th column of V. The induction hypothesis gives

$$|W| = \left(\prod_{r \geqq j > i \geqq 2} (\alpha_j - \alpha_i)^{s_j s_i}\right) \left(\prod_{j=1}^{r} (\alpha_j - \alpha_1)^{s_j (s_1 - 1)}\right),$$

(with the convention that an empty product is 1), from which we deduce the value of $|V|$. Q.E.D.

Notice also that the determinant of the matrix considered by Jarden [3] differs from $|V|$ by the factor

$$(s_1 - 1)!\,(s_2 - 1)! \ldots (s_r - 1)!\,.$$

Corollary 4. *If $f(X)$ has m distinct roots, the m sequences $\{\alpha_1^n\}_{n \in \mathbf{N}}, \{\alpha_2^n\}_{n \in \mathbf{N}}, \ldots, \{\alpha_m^n\}_{n \in \mathbf{N}}$ form a basis of $\mathscr{S}_m$.*

Proof. The Vandermonde determinant

$$\prod_{i<j} (\alpha_j - \alpha_i)$$

is $\neq 0$. Q.E.D.

Note in passing that Corollary 2 and Corollary 4 are in fact particular cases of a general theorem involving the so-called Casorati determinants, named after Felice Casorati (1835–1890): see Theorem 5.1 of [6], or see [5].

Let us finally remark that the matrix V introduced in the proof of Theorem 3 is the transpose of Kalman's so-called generalized Vandermonde matrix which has previously appeared as a special kind of confluent alternant matrix: for the proof of formula 4, see [4], chapter VI of [1], chapter 3 of [2], and [3]; however the proofs involving elementary row operations are either sketched (5 × 5 example) or left to the reader.

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Zur Abschätzung des Brocardschen Winkels, II

Für Definitionen und Bezeichnungen sei auf [1] verwiesen.
In [1] wurde gezeigt, dass

$$2\,\omega \le \frac{3}{\sum 1/\alpha_i}, \tag{1}$$

und dass Gleichheit genau im gleichseitigen Fall besteht (*GGG*). Diese Note ist dem Beweis der in [1] vermuteten Ungleichung

$$\sqrt{\frac{3}{\sum 1/\alpha_i^2}} \le 2\,\omega \qquad (GGG) \tag{2}$$

gewidmet. (1) und (2) bedeuten schon eine sehr gute Abschätzung von ω, wenn man beachtet, dass vor zwanzig Jahren nur

$$\min(\alpha_1, \alpha_2, \alpha_3) \le 2\,\omega \le \frac{\pi}{3} \qquad (GGG)$$

bekannt war. Wir werden im Satz 3 die Frage nach der besten Abschätzung genauer stellen.
Der Beweis von (2) stützt sich auf dieselbe Beweisidee wie (1). Dabei stösst man aber auf viele neue technische Schwierigkeiten. Nach zwei Lemmas werden wir zunächst (2) auf den gleichschenkligen Fall reduzieren.

Lemma 1: (i) *Die Funktion* $f(x) := \frac{1}{x} - \cot x$ *ist strikt konvex in* $(0, \pi)$.
(ii) *Es gilt*

$$\frac{1}{x} - \cot x > \frac{x}{3} > \frac{\operatorname{tg} x}{3 + \operatorname{tg}^2 x} =: g(x)$$

für $x \in \left(0, \frac{\pi}{2}\right)$.

(iii) *Die Funktion* $h(x) := \left(\frac{x}{\sin x}\right)^4 (1 + 2\cos^2 x)$ *ist strikt monoton wachsend in* $(0, \pi)$.