

# Some inequalities for the triangle

Autor(en): **Miloševi, M.**

Objektyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **42 (1987)**

Heft 5

PDF erstellt am: **13.09.2024**

Persistenter Link: <https://doi.org/10.5169/seals-40040>

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0013-6018/87/050000-00\$1.50+0.20/0

## Some inequalities for the triangle

**Notation.**  $a, b, c$  – sides BC, CA, AB of a triangle ABC;  $\alpha, \beta, \gamma$  – its angles;  $s$  – semi-perimeter;  $F$  – area;  $R$  – radius of circumcircle;  $r$  – radius of incircle;  $h_a, h_b, h_c$  – altitudes;  $m_a, m_b, m_c$  – medians and  $r_a, r_b, r_c$  – radii of excircles.

**Theorem 1.** *In every triangle the following equalities are valid:*

$$\frac{h_b + h_c}{r_a} + \frac{h_c + h_a}{r_b} + \frac{h_a + h_b}{r_c} = 6 \quad (1)$$

$$\cos^3 \frac{\alpha}{2} \cos \frac{\beta - \gamma}{2} + \cos^3 \frac{\beta}{2} \cos \frac{\gamma - \alpha}{2} + \cos^3 \frac{\gamma}{2} \cos \frac{\alpha - \beta}{2} = \frac{3s}{4R}. \quad (2)$$

**Proof.** Since

$$h_a = \frac{2F}{a}, \quad h_b = \frac{2F}{b}, \quad h_c = \frac{2F}{c} \quad \text{and}$$

$$r_a = \frac{F}{s-a}, \quad r_b = \frac{F}{s-b}, \quad r_c = \frac{F}{s-c},$$

we have

$$\begin{aligned} & \frac{h_b + h_c}{r_a} + \frac{h_c + h_a}{r_b} + \frac{h_a + h_b}{r_c} \\ &= \frac{2}{abc} (a(s-a)(b+c) + b(s-b)(c+a) + c(s-c)(a+b)) \\ &= \frac{2}{abc} (a^3 + b^3 + c^3 - 2s(a^2 + b^2 + c^2) + 2s(ab + bc + ca)). \end{aligned}$$

From  $abc = 4Rrs$ ,  $ab + bc + ca = r^2 + s^2 + 4Rr$ ,  $a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2)$  and  $a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2)$ , we get

$$\begin{aligned} & \frac{h_b + h_c}{r_a} + \frac{h_c + h_a}{r_b} + \frac{h_a + h_b}{r_c} \\ &= \frac{1}{2Rrs} (2s(r^2 + s^2 + 4Rr) - 2s \cdot 2(s^2 - 4Rr - r^2) + 2s(s^2 - 6Rr - 3r^2)) = 6, \end{aligned}$$

i.e. (1) holds.

Moreover, since

$$h_b + h_c = 8R \sin \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} \cos \frac{\beta - \gamma}{2}, \quad h_c + h_a = 8R \sin \frac{\beta}{2} \cos^2 \frac{\beta}{2} \cos \frac{\gamma - \alpha}{2},$$

$$h_a + h_b = 8R \sin \frac{\gamma}{2} \cos^2 \frac{\gamma}{2} \cos \frac{\alpha - \beta}{2} \quad \text{and}$$

$$r_a = s \cdot \tan \frac{\alpha}{2}, \quad r_b = s \cdot \tan \frac{\beta}{2}, \quad r_c = s \cdot \tan \frac{\gamma}{2},$$

we obtain

$$\begin{aligned} & \frac{h_b + h_c}{r_a} + \frac{h_c + h_a}{r_b} + \frac{h_a + h_b}{r_c} \\ &= \frac{8R}{s} \left( \cos^3 \frac{\alpha}{2} \cos \frac{\beta - \gamma}{2} + \cos^3 \frac{\beta}{2} \cos \frac{\gamma - \alpha}{2} + \cos^3 \frac{\gamma}{2} \cos \frac{\alpha - \beta}{2} \right), \end{aligned}$$

which is equivalent to

$$\cos^3 \frac{\alpha}{2} \cos \frac{\beta - \gamma}{2} + \cos^3 \frac{\beta}{2} \cos \frac{\gamma - \alpha}{2} + \cos^3 \frac{\gamma}{2} \cos \frac{\alpha - \beta}{2} = \frac{3}{4} \cdot \frac{s}{R},$$

i.e., (2).

**Corollary 1.** *In every triangle*

$$3 \cdot 2^n \leq \left( \frac{h_b + h_c}{r_a} \right)^n + \left( \frac{h_c + h_a}{r_b} \right)^n + \left( \frac{h_a + h_b}{r_c} \right)^n \leq 6^n, \quad n \geq 1. \tag{A}$$

*Equality in (A) holds if and only if the triangle is equilateral and  $n = 1$ .*

**Proof.** The inequality (A) follows from (1) putting  $x_1 = \frac{h_b + h_c}{r_a}$ ,  $x_2 = \frac{h_c + h_a}{r_b}$  and  $x_3 = \frac{h_a + h_b}{r_c}$  in ([7], 47)

$$3 \left( \sum_{i=1}^3 \frac{x_i}{3} \right)^n \leq \sum_{i=1}^3 x_i^n \leq \left( \sum_{i=1}^3 x_i \right)^n, \quad (n \geq 1, x_i > 0).$$

**Remark 1.** a) The left part of (A) holds for  $n \leq 0$ . b) If  $0 < n < 1$ , then the inequality (A) is reversed.

**Remark 2.** For  $n = -1$ , we have the inequality 6.26 in [3]:

$$\frac{r_a}{h_b + h_c} + \frac{r_b}{h_c + h_a} + \frac{r_c}{h_a + h_b} \geq \frac{3}{2}.$$

**Corollary 2.** *The following inequality*

$$\cos^3 \frac{\alpha}{2} \cos \frac{\beta - \gamma}{2} + \cos^3 \frac{\beta}{2} \cos \frac{\gamma - \alpha}{2} + \cos^3 \frac{\gamma}{2} \cos \frac{\alpha - \beta}{2} \leq \frac{9\sqrt{3}}{8}$$

*holds, with inequality only for  $\alpha = \beta = \gamma$ .*

**Proof.** This follows from (2) and  $2s = a + b + c \leq 3R\sqrt{3}$  ([3], 5.3).

**Theorem 2.** *In every triangle*

$$\left( \frac{m_a}{r_a} \right)^n + \left( \frac{m_b}{r_b} \right)^n + \left( \frac{m_c}{r_c} \right)^n \geq 3^{1-3n/2} \cdot \left( \frac{s}{r} \right)^n \geq 3, \quad n \geq 2/3, \quad (\text{B})$$

*and the equality is true if and only if the triangle is equilateral.*

**Proof.** Since ([5], 156)

$$m_a \geq \sqrt{s(s-a)}, \quad m_b \geq \sqrt{s(s-b)}, \quad m_c \geq \sqrt{s(s-c)},$$

we have

$$\left( \frac{m_a}{r_a} \right)^n + \left( \frac{m_b}{r_b} \right)^n + \left( \frac{m_c}{r_c} \right)^n \geq \frac{1}{r^n s^{n/2}} ((s-a)^{3n/2} + (s-b)^{3n/2} + (s-c)^{3n/2}).$$

Let us consider the function

$$f(x) = (s-x)^{3n/2}, \quad (0 < x < s, n \geq 2/3)$$

and its second derivative

$$f''(x) = \frac{3n}{4} (3n-2) (s-x)^{\frac{3n}{2}-2} \geq 0.$$

Hence, the function  $f$  is convex, so that

$$\sum_{i=1}^3 (s-x_i)^{3n/2} \geq 3^{1-3n/2} \left(3s - \sum_{i=1}^3 x_i\right)^{3n/2}, \quad n \geq \frac{2}{3}. \tag{4}$$

Putting  $x_1 = a$ ,  $x_2 = b$  and  $x_3 = c$  in (4), we get

$$(s-a)^{3n/2} + (s-b)^{3n/2} + (s-c)^{3n/2} \geq 3^{1-3n/2} \cdot s^{3n/2}. \tag{5}$$

Now (5) and (3) imply that

$$\left(\frac{m_a}{r_a}\right)^n + \left(\frac{m_b}{r_b}\right)^n + \left(\frac{m_c}{r_c}\right)^n \geq 3^{1-3n/2} \left(\frac{s}{r}\right)^n, \quad n \geq \frac{2}{3}. \tag{6}$$

From (6) and  $s^2 \geq 27r^2$  ([3], 5.11), we obtain:

$$\left(\frac{m_a}{r_a}\right)^n + \left(\frac{m_b}{r_b}\right)^n + \left(\frac{m_c}{r_c}\right)^n \geq 3.$$

**Theorem 3.** *In every triangle*

$$4\sqrt{3}(3R-4r) \leq \frac{a^3}{r_b r_c} + \frac{b^3}{r_c r_a} + \frac{c^3}{r_a r_b} \leq \frac{2R\sqrt{3}}{r}(3R-4r). \tag{C}$$

*Equality occurs if and only if the triangle is equilateral.*

**Proof.** We have

$$\begin{aligned} & \frac{a^3}{r_b r_c} + \frac{b^3}{r_c r_a} + \frac{c^3}{r_a r_b} \\ &= \frac{1}{F^2} (a^3 (s-b)(s-c) + b^3 (s-c)(s-a) + c^3 (s-a)(s-b)) \\ &= \frac{1}{F^2} (s(a^4 + b^4 + c^4) - s^2(a^3 + b^3 + c^3) + abc(a^2 + b^2 + c^2)). \end{aligned}$$

From  $F = rs$ ,  $abc = 4Rrs$ ,  $a^2 + b^2 + c^2 = 2 \cdot (s^2 - 4Rr - r^2)$ ,  $a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2)$  and  $a^4 + b^4 + c^4 = 2(s^2 - 4Rr - r^2)^2 - 2(2rs)^2$ , we obtain

$$\begin{aligned} \frac{a^3}{r_b r_c} + \frac{b^3}{r_c r_a} + \frac{c^3}{r_a r_b} &= \frac{2}{rs} (4Rr^2 + 2Rs^2 - 3rs^2 + r^3) \\ &= \frac{2}{F} (r \cdot r(4R+r) + s \cdot s(2R-3r)). \end{aligned} \tag{7}$$

Then, since ([3], 7.2, 5.11)

$$r(4R+r) \geq F\sqrt{3}$$

and

$$s^2 \geq 27r^2,$$

(7) implies

$$\frac{a^3}{r_b r_c} + \frac{b^3}{r_c r_a} + \frac{c^3}{r_a r_b} \geq \frac{2}{F} (r \cdot F\sqrt{3} + s \cdot 3r\sqrt{3}(2R-3r)) = 4\sqrt{3}(3R-4r),$$

i. e. the first part of the inequality (C).

By 5.3 and 5.6 in [3] and (7) we get the second part of (C).

**Theorem 4.** *In every triangle*

$$\frac{4\sqrt{3}}{3} \cdot \frac{7R-2r}{9R-2r} \leq \frac{a}{h_b+h_c} + \frac{b}{h_c+h_a} + \frac{c}{h_a+h_b} \leq \sqrt{\frac{3R}{2r}}. \quad (\text{D})$$

*Equality holds if and only if the triangle is equilateral.*

**Proof.** Since

$$h_a = \frac{2F}{a}, \quad h_b = \frac{2F}{b}, \quad h_c = \frac{2F}{c},$$

we have

$$\begin{aligned} \frac{a}{h_b+h_c} + \frac{b}{h_c+h_a} + \frac{c}{h_a+h_b} &= \frac{abc}{2F} \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) \\ &= \frac{abc}{2F} \cdot \frac{(a^2+b^2+c^2+3(ab+bc+ca))}{((a+b+c)(ab+bc+ca)-abc)}. \end{aligned}$$

From  $abc = 4Rrs$ ,  $F = rs$ ,  $a+b+c = 2s$ ,  $ab+bc+ca = r^2+s^2+4Rr$  and  $a^2+b^2+c^2 = 2(s^2-4Rr-r^2)$ , we get

$$\begin{aligned} \frac{a}{h_b+h_c} + \frac{b}{h_c+h_a} + \frac{c}{h_a+h_b} &= \frac{R}{s} \cdot \frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2} \\ &= \frac{R}{s} \left( 5 - \frac{2r(3R+2r)}{s^2+2Rr+r^2} \right). \end{aligned}$$

Using ([3], 5.8, 5.3)

$$s^2 \geq r(16R - 5r)$$

and

$$2s = a + b + c \leq 3R\sqrt{3},$$

we obtain

$$\begin{aligned} \frac{a}{h_b + h_c} + \frac{b}{h_c + h_a} + \frac{c}{h_a + h_b} &\geq \frac{R}{s} \left( 5 - \frac{3R + 2r}{9R - 2r} \right) \\ &= \frac{R}{s} \cdot \frac{6(7R - 2r)}{9R - 2r} \\ &\geq \frac{4\sqrt{3}}{3} \cdot \frac{7R - 2r}{9R - 2r}, \end{aligned}$$

i. e. the left part of the inequality (D).

From the inequalities

$$h_b + h_c \geq 2\sqrt{h_b h_c}, \quad h_c + h_a \geq 2\sqrt{h_c h_a}, \quad h_a + h_b \geq 2\sqrt{h_a h_b},$$

it follows that

$$\begin{aligned} \frac{a}{h_b + h_c} + \frac{b}{h_c + h_a} + \frac{c}{h_a + h_b} &= 2F \left( \frac{1}{h_a(h_b + h_c)} + \frac{1}{h_b(h_c + h_a)} + \frac{1}{h_c(h_a + h_b)} \right) \\ &\leq \frac{F}{\sqrt{h_a h_b h_c}} \left( \frac{1}{\sqrt{h_a}} + \frac{1}{\sqrt{h_b}} + \frac{1}{\sqrt{h_c}} \right). \end{aligned} \tag{8}$$

If in the well-known inequality

$$(x + y + z)^2 \leq 3(x^2 + y^2 + z^2), \quad (x, y, z > 0), \tag{9}$$

we put  $x = \frac{1}{\sqrt{h_a}}$ ,  $y = \frac{1}{\sqrt{h_b}}$  and  $z = \frac{1}{\sqrt{h_c}}$ , we obtain

$$\left( \frac{1}{\sqrt{h_a}} + \frac{1}{\sqrt{h_b}} + \frac{1}{\sqrt{h_c}} \right)^2 \leq 3 \left( \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \right). \tag{10}$$

From  $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$ ,  $h_a h_b h_c = \frac{2F^2}{R}$ , (8) and (10), we get

$$\frac{a}{h_b + h_c} + \frac{b}{h_c + h_a} + \frac{c}{h_a + h_b} \leq \frac{F}{\sqrt{h_a h_b h_c}} \sqrt{\frac{3}{r}} = \sqrt{\frac{3R}{2r}}.$$

This proves the right part of (D).

**Remark 4.** From (D) using the inequality 5.1 in [3] we have the inequality

$$\sqrt{3} \leq \frac{a}{h_b+h_c} + \frac{b}{h_c+h_a} + \frac{c}{h_a+h_b} \leq \frac{R}{2r} \sqrt{3},$$

shown in [4].

**Theorem 5.** *In every triangle*

$$\frac{m_a^2}{bc} + \frac{m_b^2}{ca} + \frac{m_c^2}{ab} \leq \frac{5}{4} + \frac{R}{2r}, \quad (\text{E})$$

*with equality holding if and only if the triangle is equilateral.*

**Proof.** Using Bager's identity ([2], 39)

$$am_a^2 + bm_b^2 + cm_c^2 = \frac{s}{2} (s^2 + 2Rr + 5r^2),$$

we get

$$\begin{aligned} \frac{m_a^2}{bc} + \frac{m_b^2}{ca} + \frac{m_c^2}{ab} &= \frac{1}{abc} (am_a^2 + bm_b^2 + cm_c^2) \\ &= \frac{s}{2abc} (s^2 + 2Rr + 5r^2). \end{aligned}$$

Then, since ([3], 5.8, 5.1)

$$s^2 \leq 4R^2 + 4Rr + 3r^2$$

and

$$2r \leq R,$$

we have

$$\begin{aligned} \frac{m_a^2}{bc} + \frac{m_b^2}{ca} + \frac{m_c^2}{ab} &\leq \frac{1}{8Rr} (4R^2 + 6Rr + 8r^2) \\ &\leq \frac{1}{8Rr} \left( 4R^2 + 6Rr + 8r \cdot \frac{R}{2} \right) \\ &= \frac{5}{4} + \frac{R}{2r}, \end{aligned}$$

i. e. (E) holds.



**Theorem 6.** *For acute triangle the following inequality holds*

$$a^2 \tan \alpha + b^2 \tan \beta + c^2 \tan \gamma \geq \frac{4 r s^3}{2 R^2 + r^2}. \tag{F}$$

**Proof.** If in Jensen's inequality for a convex function

$$f\left(\frac{\sum_{i=1}^3 p_i}{\sum_{i=1}^3 x_i}\right) \leq \frac{\sum_{i=1}^3 x_i f\left(\frac{p_i}{x_i}\right)}{\sum_{i=1}^3 x_i}, \quad p_i \text{ and } \frac{p_i}{x_i} \in \bar{I},$$

we put

$$x_1 = a, \quad x_2 = b, \quad x_3 = c, \quad p_1 = a^2, \quad p_2 = b^2 \quad \text{and} \quad p_3 = c^2,$$

then, for  $f(x) = 1/x$ , from ([1], 43; [3] 5.14)

$$a^2 + b^2 + c^2 = 4 F (\cot \alpha + \cot \beta + \cot \gamma)$$

and

$$a^2 + b^2 + c^2 \leq 8 R^2 + 4 r^2,$$

we obtain (F).

**Remark 5.** This method shown in [6].

**Theorem 7.** *In every triangle*

$$3 \sqrt{3 r} \leq \sqrt{h_a} + \sqrt{h_b} + \sqrt{h_c} \leq \left(1 + \frac{r}{R}\right) \sqrt{6 R}. \tag{G}$$

*Equality holds if and only if the triangle is equilateral.*

**Proof.** By means of the arithmetic-geometric inequality, we get

$$\begin{aligned} \sqrt{h_a} + \sqrt{h_b} + \sqrt{h_c} &\geq 3 \sqrt[6]{\frac{2 F}{a} \cdot \frac{2 F}{b} \cdot \frac{2 F}{c}} \\ &= 3 \sqrt[6]{\frac{2 r^2 s^2}{R}}. \end{aligned} \tag{11}$$

Then, since ([3], 5.12)

$$2 s^2 \geq 27 R r,$$

(11) implies

$$\sqrt{h_a} + \sqrt{h_b} + \sqrt{h_c} \geq 3\sqrt{3r},$$

i. e. the left part of the inequality (G).

If in the inequality (9) we put  $x = \frac{1}{\sqrt{a}}$ ,  $y = \frac{1}{\sqrt{b}}$  and  $z = \frac{1}{\sqrt{c}}$ , then, from ([3], 5.17)

$$ab + bc + ca \leq 4(R+r)^2,$$

we obtain

$$\begin{aligned} \sqrt{h_a} + \sqrt{h_b} + \sqrt{h_c} &\leq \sqrt{2F} \cdot \sqrt{3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} \\ &= \sqrt{\frac{3}{2R}(ab + bc + ca)} \\ &\leq \left(1 + \frac{r}{R}\right) \sqrt{6R}, \end{aligned}$$

i. e. the right part of (G).

**Remark 6.** The second part of the inequality (G) is more precise than

$$\sqrt{h_a} + \sqrt{h_b} + \sqrt{h_c} \leq \frac{3}{2} \sqrt{6R},$$

shown in [3].

**Theorem 8.** *In every triangle the following inequality is valid:*

$$\frac{r_b r_c}{a^2} + \frac{r_c r_a}{b^2} + \frac{r_a r_b}{c^2} \geq \frac{9}{4}. \quad (\text{H})$$

*Equality holds if and only if the triangle is equilateral.*

**Proof 1.** Inequality (H) is equivalent to

$$\frac{s-a}{a^2} + \frac{s-b}{b^2} + \frac{s-c}{c^2} \geq \frac{9}{4s}. \quad (12)$$

The function

$$f(x) = \frac{s}{x^2} - \frac{1}{x} \quad (0 < x < s)$$

is convex, so that

$$\sum_{i=1}^3 f(x_i) \geq 3 \cdot f\left(\frac{\sum_{i=1}^3 x_i}{3}\right), \quad x_i \in \bar{I}.$$

If we put  $x_1 = a$ ,  $x_2 = b$  and  $x_3 = c$ , we get the inequality (H).

**Proof 2.** As

$$\begin{aligned} & \frac{r_b r_c}{a^2} + \frac{r_c r_a}{b^2} + \frac{r_a r_b}{c^2} \\ &= \frac{s(s-a)}{a^2} + \frac{s(s-b)}{b^2} + \frac{s(s-c)}{c^2} \\ &= \frac{s^2}{a^2 b^2 c^2} (b^2 c^2 + c^2 a^2 + a^2 b^2) - \frac{s}{a b c} (a b + b c + c a) \\ &= \frac{1}{16 R^2 r^2} (b^2 c^2 + c^2 a^2 + a^2 b^2 - 4 R r (a b + b c + c a)). \end{aligned} \tag{13}$$

From ([3], 5.17)

$$\begin{aligned} a b + b c + c a &\geq 4 r (5 R - r), \\ (b c + a b + c a)^2 &\leq 3 (b^2 c^2 + c^2 a^2 + a^2 b^2) \end{aligned}$$

and (13), we have

$$\begin{aligned} \frac{r_b r_c}{a^2} + \frac{r_c r_a}{b^2} + \frac{r_a r_b}{c^2} &\geq \frac{a b + b c + c a}{48 R^2 r^2} (a b + b c + c a - 12 R r) \\ &\geq \frac{(5 R - r) (2 R - r)}{3 R^2}. \end{aligned} \tag{14}$$

By (14) and the inequality 5.1 in [3], we obtain (H).

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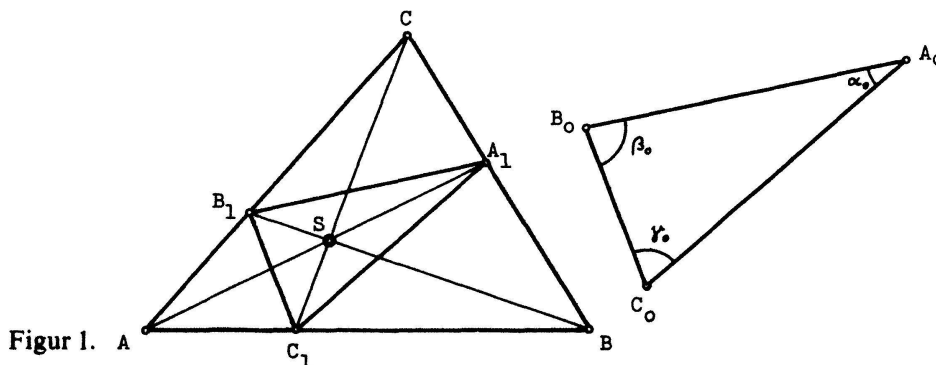
## Ceva-Dreiecke

Ein in einem Dreieck  $ABC$  einbeschriebenes Dreieck  $A_1B_1C_1$  (Fig. 1) möge *Ceva-Dreieck* heissen, wenn sich die drei Transversalen  $AA_1$ ,  $BB_1$ ,  $CC_1$  in einem Punkt  $S$  schneiden. Nach dem Satz von Ceva tritt dies genau dann ein, wenn die drei Teilverhältnisse

$$[ABC_1] = -\tau < 0; \quad [BCA_1] = -\varrho < 0; \quad [CAB_1] = -\sigma < 0 \quad (1)$$

der Gleichung genügen

$$\varrho \cdot \sigma \cdot \tau = 1. \quad (2)$$



Figur 1. A

Im folgenden soll gezeigt werden:

**Satz 1:** Gegeben sei ein beliebiges Dreieck  $ABC$  und ein ebenfalls beliebiges Dreieck  $A_0B_0C_0$  (Fig. 1). Dann gibt es genau ein dem Dreieck  $ABC$  einbeschriebenes Ceva-Dreieck  $A_1B_1C_1$ , das dem Dreieck  $A_0B_0C_0$  ähnlich ist.

**Beweis:** Zunächst weisen wir die Existenz eines solchen Dreiecks  $A_1B_1C_1$  nach. Dazu führen wir ein Koordinatensystem ein, das dem vorgegebenen Dreieck  $ABC$  nach Fig. 2 angepasst ist.

Aus der Figur können die Bezeichnungen für die Masszahlen der Winkel (im Bogenmass) sowie die Koordinaten der Punkte (in Abhängigkeit von den Teilverhältnissen  $\varrho$ ,  $\sigma$ ,  $\tau$ ) abgelesen werden. Dabei dürfen wir ohne Einschränkung der Allgemeinheit voraussetzen, dass

$$b, c, b-a \in \mathbf{R}^+; \quad a \in \mathbf{R}; \quad \alpha_1, \beta_1, \gamma_1 \in \mathbf{R}^+; \quad \alpha_1 + \beta_1 + \gamma_1 = \pi. \quad (3)$$