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A conjecture of B. Sendov, better known as Ilieff's conjecture, states that if all zeros of a polynomial $P(z)$ lie in the unit disc $|z| \leq 1$ and if z_0 is any one such zero, then the disc $|z - z_0| \leqq 1$ contains at least one zero of $P'(z)$. Proofs of this conjecture have been given for polynomials of degree at most five and in a number of other special cases for polynomials of any degree. In particular, proofs of Ilieff's conjecture for polynomials with three zeros have been given by Saff and Twomey [5], Schmeisser [6] and Cohen and Smith [1]. References to other special cases may be found in Marden [2] and Schmeisser [6].

The purpose of this note is to prove that Ilieff's conjecture is true for all polynomials with at most four zeros. The methods we use are elementary and the paper is fully self-contained.

We set

$$P(z) = \prod_{j=0}^{p-1} (z - z_j)^{n_j}$$

where the n_j are arbitrary positive integers and the z_j are distinct with $|z_j| \leqq 1$, $j = 0, 1, \ldots, p-1$. We wish to show that $P'(z)$ has a zero in $|z - z_0| \leqq 1$ if $p \leqq 4$. This is immediate if $n_0 > 1$, so suppose $n_0 = 1$. Without loss of generality, we may assume z_0 to be real with $0 \leqq z_0 \leqq 1$.

Let us put

$$P(z) = (z - z_0) \prod_{j=1}^{p-1} (z - z_j)^{n_j} = (z - z_0)\, Q(z)\,.$$

Let $n = 1 + n_1 + n_2 + \ldots + n_{p-1}$. Then if $w_1, \ldots, w_{n-1}$ are the zeros of $P'(z)$, we have

$$P'(z) = n \prod_{j=1}^{n-1} (z - w_j)\,.$$

By suitable relabelling of the w_j we can write

$$P'(z_0) = n \prod_{j=1}^{p-1} (z_0 - \zeta_j)(z_0 - z_j)^{n_j - 1}\,.$$

On the other hand, we have

$$P'(z_0) = Q(z_0) = \prod_{j=1}^{p-1} (z_0 - z_j)^{n_j}\,.$$

Writing $r_j = |z_0 - z_j|$, $\varrho_j = |z_0 - \zeta_j|$ $(j = 1, 2, \ldots, p-1)$ and comparing the above two expressions for $P'(z_0)$ gives the result

$$r_1 r_2 \ldots r_{p-1} = n \varrho_1 \varrho_2 \ldots \varrho_{p-1}. \tag{1}$$

It is clear that $0 < r_j \leqq 2$ $(j = 1, \ldots, p-1)$.

Remark 1. It follows from Equation (1) that Ilieff's conjecture is true when $r_1 r_2 \ldots r_{p-1} \leqq n$, which will certainly be the case if $z_0 = 0$ or if $n \geqq 2^{p-1}$. In particular, Ilieff's conjecture is true if $p = 2$, and if $p = 3$ and the polynomial has a multiple zero.

Lemma 1. *If* $|P''(z_0)| \geqq (n-1)|P'(z_0)|$ *then there exists a zero* w *of* $P'(z)$ *such that* $|z_0 - w| \leqq 1$.

Proof. Suppose $|z_0 - w_j| > 1$ for $j = 1, \ldots, n-1$. We recall that z_0 is a simple zero of $P(z)$. Then

$$\left|\frac{P''(z_0)}{P'(z_0)}\right| = \left|\sum_{j=1}^{n-1} \frac{1}{z_0 - w_j}\right| \leqq \sum_{j=1}^{n-1} \frac{1}{|z_0 - w_j|} < n-1.$$

Lemma 2. *If* $\sum_{j=1}^{p-1} n_j/r_j^2 \leqq (n-1)/(1+z_0)$, *then there exists a zero* w *of* $P'(z)$ *such that* $|z_0 - w| \leqq 1$.

Proof. Since the zero z_0 is simple,

$$\left|\frac{P''(z_0)}{P'(z_0)}\right| \geqq \Re\left(\frac{P''(z_0)}{P'(z_0)}\right) = \Re\left(\frac{2\,Q'(z_0)}{Q(z_0)}\right) = 2\sum_{j=1}^{p-1} \Re\left(\frac{n_j}{z_0 - z_j}\right).$$

We can show by elementary properties of complex numbers that for any real $\alpha > 0$ and any z with $|z| \leqq 1$, $z \neq \alpha$, we have

$$\Re\left(\frac{1}{\alpha - z}\right) \geqq \frac{1}{2\alpha} - \frac{1-\alpha^2}{2\alpha}\,\frac{1}{|\alpha - z|^2}.$$

(See also [3].) By Remark 1, we may assume $z_0 > 0$. Hence

$$\begin{aligned}\left|\frac{P''(z_0)}{P'(z_0)}\right| &\geqq 2\sum_{j=1}^{p-1} n_j\left(\frac{1}{2z_0} - \frac{1-z_0^2}{2z_0}\,\frac{1}{r_j^2}\right)\\ &= \frac{n-1}{z_0} - \frac{1-z_0^2}{z_0}\sum_{j=1}^{p-1}\frac{n_j}{r_j^2} \qquad (2)\\ &\geqq n-1\end{aligned}$$

if $\sum_{j=1}^{p-1} n_j/r_j^2 \leqq (n-1)/(1+z_0)$. The result now follows by Lemma 1.

Remark 2. 1. If $z_0 = 1$ in (2) above, then $|P''(z_0)| \geqq (n-1)\,|P'(z_0)|$ and Lemma 1 implies that there is a zero of $P'(z)$ in $|z - z_0| \leqq 1$. This is another proof of the boundary case (cf. [4]).
2. Suppose $n = p = 3$, and assume, because of Remark 1, that $r_1 r_2 > 3$. Then

$$\sum_{j=1}^{p-1} \frac{n_j}{r_j^2} = \frac{1}{r_1^2} + \frac{1}{r_2^2} < \frac{4}{9} + \frac{1}{3} < 1 .$$

Using Lemma 2, this provides a new and very simple proof that Ilieff's conjecture is true for polynomials of degree 3. In conjunction with Remark 1, we see that the conjecture is true for polynomials with two or three distinct zeros. A few more simple lemmas will allow us to show it to be true for polynomials with four zeros.

Lemma 3. *For a fixed integer $n \geqq 4$ the function given by*

$$f(x) = \frac{4(n-3)\,x^2}{n^2} + \frac{1}{x^2}$$

is an increasing function of x when $x > \sqrt{n/2}$.

Proof. We have $f'(x) > 0$ if $x > (n-3)^{-1/4}\sqrt{n/2}$, and the result follows.

Lemma 4. *For a fixed integer n, $4 \leqq n \leqq 8$, set*

$$g(x) = \frac{4(n-3)\,x^2}{n^2} + \frac{2}{x^2} .$$

Then $g(x) \leqq (n-1)/2$ if $n^{1/3} \leqq x \leqq 2$.

Proof. Suppose $n = 4$. The only positive root of $g'(x) = 0$ is $x = 2^{3/4}$ and since $g''(x) > 0$ for $x > 0$, $g(x)$ has a minimum at $x = 2^{3/4}$. The result follows since $g(4^{1/3}) < g(2) = 3/2$.
Now suppose $n \geqq 5$. We find that $g'(x) \geqq 0$ if $x \geqq n^{1/2}\,2^{-1/4}(n-3)^{-1/4} = x_0$, say. The result follows since $x_0 < n^{1/3}$ and $g(2) < (n-1)/2$.

We can now give our main result.

Theorem. *Let $P(z) = (z - z_0)\prod_{j=1}^{3}(z - z_j)^{n_j}$, where z_0, z_1, z_2, z_3 are distinct and $|z_j| \leqq 1$, $j = 0, 1, 2, 3$. Then $P'(z)$ has a zero w such that $|z_0 - w| \leqq 1$.*

Proof. As above, we may assume that z_0 is real and $0 < z_0 < 1$.

From Equation (1), if $r_1 r_2 r_3 \leqq n$, then $\varrho_1 \varrho_2 \varrho_3 \leqq 1$, and the result follows. Assume then that $4 \leqq n < r_1 r_2 r_3 \leqq 8$, $0 < r_1 \leqq r_2 \leqq r_3 \leqq 2$, from which we have

$$r_1 r_2 > n/2, \quad r_3 > n^{1/3}, \quad r_2 > \sqrt{n/2} .$$

Then

$$\frac{n_1}{r_1^2}+\frac{n_2}{r_2^2}+\frac{n_3}{r_3^2} \leqq \frac{n-3}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2} < \frac{4r_2^2(n-3)}{n^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}$$

$$= f(r_2)+\frac{1}{r_3^2} \qquad \leqq f(r_3)+\frac{1}{r_3^2} = g(r_3) \leqq \frac{n-1}{2}.$$

(by Lemma 3) (by Lemma 4)

The result follows by application of Lemma 2.

Remark 3. In view of Lemma 1, it is interesting to note also the following result.

Let $P(z)$ be a polynomial of degree n, all of whose zeros lie in $|z| \leqq 1$; let z_0 be a simple zero of $P(z)$ and assume z_0 is real and positive. If $|P''(z_0)| \leqq (n-1)\, z_0 \,|P'(z_0)|/2$ then there exists a zero w of $P'(z)$ such that $|z_0 - w| \leqq 1$.

The proof uses the inequality in the proof of Lemma 2, and also the Gauss-Lucas theorem, which states that the zeros of $P'(z)$ lie in the closed convex hull of the zeros of $P(z)$. If $w_1, \ldots, w_{n-1}$ are the zeros of $P'(z)$, then $|w_j| \leqq 1$ for $j = 1, \ldots, n-1$. Suppose $|z_0 - w_j| > 1$ for $j = 1, \ldots, n-1$. We have

$$\left|\frac{P''(z_0)}{P'(z_0)}\right| = \left|\sum_{j=1}^{n-1} \frac{1}{z_0 - w_j}\right| \geqq \Re\left(\sum_{j=1}^{n-1} \frac{1}{z_0 - w_j}\right)$$

$$\geqq \sum_{j=1}^{n-1}\left(\frac{1}{2z_0} - \frac{1-z_0^2}{2z_0}\,\frac{1}{|z_0 - w_j|^2}\right)$$

$$= \frac{n-1}{2z_0} - \frac{1-z_0^2}{2z_0}\sum_{j=1}^{n-1}\frac{1}{|z_0 - w_j|^2} > \frac{(n-1)\,z_0}{2}.$$

The desired result then follows.

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