Generalization of a formula of C. Buchta about the convex hull of random points

1. Introduction

For an arbitrary convex body K in the d-dimensional Euclidean space $E^d (d \geqq 2)$ we denote by $V_n^{(d)}(K)$ the expected volume of the convex hull H_n of n random points chosen independently and uniformly inside K. The special case $V_3^{(2)}(K)$ is directly related to a classical problem of geometrical probability, known as «Sylvester's problem»:

> What is the probability $p(K)$ that 4 points chosen identically and uniformly at random from the interior of a plane convex set K form a convex quadrilateral?

It is easy to see [6], pp. 63–64, that

$$p(K) = 1 - \frac{4}{F(K)} V_3^{(2)}(K), \tag{1.1}$$

where $F(K)$ is the area of K.
For further results and a complete list of references concerning the convex hull of random points and Sylvester's problem, see the recent survey of Buchta [2].
For arbitrary plane convex sets, respectively three-dimensional convex bodies, Buchta [1] proves the following relationships

$$V_4^{(2)}(K) = 2\,V_3^{(2)}(K) \tag{1.2}$$

and

$$V_5^{(3)}(K) = \tfrac{5}{2}\,V_4^{(3)}(K). \tag{1.3}$$

The aim of this note is to generalize Buchta's formulas (1.2) and (1.3) in the following sense:

Theorem 1. Let K be an arbitrary plane convex set. Then

$$V_{2m}^{(2)}(K) = \sum_{k=1}^{m-1} \alpha_{2m-2k+1}\, V_{2m-2k+1}^{(2)}(K) \qquad m = 2, 3, \ldots, \tag{1.4}$$

where $\alpha_{2m-2k+1}$ are constants defined by the recursion formula

$$\alpha_{2m-1} = m, \tag{1.4'}$$

$$\alpha_{2m-2k+1} = \frac{m}{2m-2k+1}\left[\binom{2m-1}{2m-2k} - \sum_{i=1}^{k-1} \frac{2m-2i+1}{2m}\binom{2m-2i}{2m-2k}\alpha_{2m-2i+1}\right],$$

$$\text{for } k = 2, 3, \ldots, m-1. \tag{1.4''}$$

Theorem 2. Let K be an arbitrary three-dimensional convex body. Then

$$V^{(3)}_{2m+1}(K) = \sum_{k=1}^{m-1} \beta_{2m-2k+2} V^{(3)}_{2m-2k+2}(K) \qquad m = 2,3,\ldots, \tag{1.5}$$

where $\beta_{2m-2k+2}$ are constants defined by the recursion formula

$$\beta_{2m} = \frac{2m+1}{2}, \tag{1.5'}$$

$$\beta_{2m-2k+2} = \frac{m(2m+1)}{(2m-2k+1)(2m-2k+2)} \cdot \left[\binom{2m-1}{2m-2k} - \sum_{i=1}^{k-1} \frac{(2m-2i+1)(2m-2i+2)}{2m(2m+1)} \binom{2m-2i}{2m-2k} \beta_{2m-2i+2}\right]$$

for $k = 2,3,\ldots,m-1$. (1.5″)

2. Proof of Theorem 1

We can assume that $F(K) = 1$ since, under an affine transformation, the expected area $V_n^{(2)}(K)$ changes only through $F(K)$.
Rényi and Sulanke [5], p. 76, show that

$$E^{(2)}_{n+1}(K) = \binom{n+1}{2} \int_K \int_K [\tilde{F}^{n-1} + (1-\tilde{F})^{n-1}]\, dP_1\, dP_2, \tag{2.1}$$

where $E^{(2)}_{n+1}(K)$ is the expected number of vertices of the convex hull H_{n+1} of $n+1$ random points $P_1, P_2, \ldots, P_{n+1}$ chosen independently and uniformly inside K, and $\tilde{F} = \tilde{F}(P_1, P_2)$ denotes the area of the smaller of the two parts of K cut off by the line through P_1 and P_2.
Efron [4], p. 335, relates the expected volume $V_n^{(d)}(K)$ of the convex hull H_n to the expected number of vertices $E^{(d)}_{n+1}(K)$ of the convex hull H_{n+1} by the following formula

$$V_n^{(d)}(K) = 1 - \frac{1}{n+1} E^{(d)}_{n+1}(K). \tag{2.2}$$

Using (2.2) expression (2.1) becomes

$$V_n^{(2)}(K) = 1 - \frac{n}{2} \int_K \int_K [\tilde{F}^{n-1} + (1-\tilde{F})^{n-1}]\, dP_1\, dP_2. \tag{2.3}$$

Consider now our formula (1.4)

$$V^{(2)}_{2m}(K) = \alpha_{2m-1} V^{(2)}_{2m-1}(K) + \ldots + \alpha_{2m-2k+1} V^{(2)}_{2m-2k+1}(K) + \ldots + \alpha_3 V_3^{(2)}(K). \tag{2.4}$$

Taking into account (2.3) we obtain

$$1 - m \int_K \int_K [\tilde{F}^{2m-1} + (1-\tilde{F})^{2m-1}]\, dP_1\, dP_2 =$$
$$= \alpha_{2m-1}\left(1 - \frac{2m-1}{2} \int_K \int_K [\tilde{F}^{2m-2} + (1-\tilde{F})^{2m-2}]\, dP_1\, dP_2\right) +$$
$$+ \ldots + \alpha_{2m-2k+1}\left(1 - \frac{2m-2k+1}{2} \int_K \int_K [\tilde{F}^{2m-2k} + (1-\tilde{F})^{2m-2k}]\, dP_1\, dP_2\right) +$$
$$+ \ldots + \alpha_3\left(1 - \frac{3}{2} \int_K \int_K [\tilde{F}^2 + (1-\tilde{F})^2]\, dP_1\, dP_2\right), \tag{2.5}$$

or, by developing the integrands,

$$1 - m \int_K \int_K \left[\sum_{i=0}^{2m-2} \binom{2m-1}{i} (-1)^i \tilde{F}^i\right] dP_1\, dP_2 =$$
$$= \alpha_{2m-1}\left(1 - (2m-1) \int_K \int_K \tilde{F}^{2m-2}\, dP_1\, dP_2 - \right.$$
$$\left. - \frac{2m-1}{2} \int_K \int_K \left[\sum_{i=0}^{2m-3} \binom{2m-2}{i} (-1)^i \tilde{F}^i\right] dP_1\, dP_2\right) +$$
$$+ \ldots + \alpha_{2m-2k+1}\left(1 - (2m-2k+1) \int_K \int_K \tilde{F}^{2m-2k}\, dP_1\, dP_2 - \right.$$
$$\left. - \frac{2m-2k+1}{2} \int_K \int_K \left[\sum_{i=0}^{2m-2k-1} \binom{2m-2k}{i} (-1)^i \tilde{F}^i\right] dP_1\, dP_2\right) +$$
$$+ \ldots + \alpha_3\left(1 - 3 \int_K \int_K \tilde{F}^2\, dP_1\, dP_2 - \frac{3}{2} \int_K \int_K [-2\tilde{F} + 1]\, dP_1\, dP_2\right). \tag{2.6}$$

Comparing the coefficients of $\int_K \int_K \tilde{F}^i\, dP_1\, dP_2$ $(i = 0, 1, \ldots, 2m-2)$ in (2.6) we get

$$m\binom{2m-1}{2m-2} = (2m-1)\,\alpha_{2m-1}\,, \tag{2.7}$$

$$m\binom{2m-1}{2m-2k} = \frac{2m-1}{2}\binom{2m-2}{2m-2k}\alpha_{2m-1} + \ldots + (2m-2k+1)\,\alpha_{2m-2k+1}\,, \quad 2 \leqq k \leqq m-1\,, \tag{2.8}$$

$$m\binom{2m-1}{2m-2k-1} = \frac{2m-1}{2}\binom{2m-2}{2m-2k-1}\alpha_{2m-1} +$$
$$+ \ldots + \frac{2m-2k+1}{2}\binom{2m-2k}{2m-2k-1}\alpha_{2m-2k+1}\,, \quad 1 \leqq k \leqq m-1 \tag{2.9}$$

and

$$m-1=\frac{2m-3}{2}\alpha_{2m-1}+\ldots+\frac{2m-2k+1}{2}\alpha_{2m-2k+3}+\ldots+\frac{1}{2}\alpha_3\,. \tag{2.10}$$

From (2.7) and (2.8) we easily conclude that

$$\alpha_{2m-1}=m \tag{2.11}$$

and

$$\alpha_{2m-2k+1}=\frac{m}{2m-2k+1}\left[\binom{2m-1}{2m-2k}-\sum_{i=1}^{k-1}\frac{2m-2i+1}{2m}\binom{2m-2i}{2m-2k}\alpha_{2m-2i+1}\right],\quad 2\leqq k\leqq m-1\,, \tag{2.12}$$

in accordance with (1.4′) and (1.4″).
Finally, expressions (2.9) and (2.10) can be verified by an adequate addition of the constants defined by (2.12).

3. Proof of Theorem 2

Analogously to the proof of Theorem 1 we can assume the volume $V(K)$ of the convex body K to be 1 since, under an affine transformation, the expected volume $V_n^{(3)}(K)$ changes only through $V(K)$.
Rényi and Sulanke's integral expression (2.1) can be extended to higher dimensions. For the three-dimensional case we can state

$$F_{n+1}^{(3)}(K)=\binom{n+1}{3}\int\limits_K\int\limits_K\int\limits_K[\tilde{V}^{n-2}+(1-\tilde{V})^{n-2}]\,dP_1\,dP_2\,dP_3\,, \tag{3.1}$$

where $F_{n+1}^{(3)}(K)$ is the expected number of faces of the convex hull H_{n+1} and $\tilde{V}=\tilde{V}(P_1,P_2,P_3)$ denotes the volume of the smaller of the two parts of K cut off by the plane through P_1, P_2 and P_3 (e.g. Buchta [1], p. 155).
With probability 1 all faces of H_{n+1} will be triangles. Therefore, taking into account expression (3.1), Efron's formula (2.2) and Euler's theorem, we conclude

$$V_n^{(3)}(K)=1-\frac{2}{n+1}-\frac{(n-1)n}{12}\int\limits_K\int\limits_K\int\limits_K[\tilde{V}^{n-2}+(1-\tilde{V})^{n-2}]\,dP_1\,dP_2\,dP_3\,. \tag{3.2}$$

To prove Theorem 2 we use the same procedure as above. We now consider formula (1.5)

$$V_{2m+1}^{(3)}(K)=\beta_{2m}V_{2m}^{(3)}(K)+\ldots+\beta_{2m-2k+2}V_{2m-2k+2}^{(3)}(K)+\ldots+\beta_4V_4^{(3)}(K)\,. \tag{3.3}$$

Taking into account expression (3.2) we obtain

$$1-\frac{2}{2m+2}-\frac{2m(2m+1)}{12}\iiint_{KKK}[\tilde{V}^{2m-1}+(1-\tilde{V})^{2m-1}]\,dP_1\,dP_2\,dP_3=$$

$$=\beta_{2m}\left(1-\frac{2}{2m+1}-\frac{(2m-1)2m}{12}\iiint_{KKK}[\tilde{V}^{2m-2}+(1-\tilde{V})^{2m-2}]\,dP_1\,dP_2\,dP_3\right)+$$

$$+\ldots+\beta_{2m-2k+2}\left(1-\frac{2}{2m-2k+3}-\frac{(2m-2k+1)(2m-2k+2)}{12}\cdot\right.$$

$$\left.\cdot\iiint_{KKK}[\tilde{V}^{2m-2k}+(1-\tilde{V})^{2m-2k}]\,dP_1\,dP_2\,dP_3\right)+$$

$$+\ldots+\beta_4\left(1-\frac{2}{5}-\frac{3\cdot 4}{12}\iiint_{KKK}[\tilde{V}^2+(1-\tilde{V})^2]\,dP_1\,dP_2\,dP_3\right), \tag{3.4}$$

or, by developing the integrants,

$$1-\frac{2}{2m+2}-\frac{2m(2m+1)}{12}\iiint_{KKK}\left[\sum_{i=0}^{2m-2}\binom{2m-1}{i}(-1)^i\tilde{V}^i\right]dP_1\,dP_2\,dP_3=$$

$$=\beta_{2m}\left(1-\frac{2}{2m+1}-\frac{2(2m-1)2m}{12}\iiint_{KKK}\tilde{V}^{2m-2}\,dP_1\,dP_2\,dP_3-\right.$$

$$\left.-\frac{(2m-1)2m}{12}\iiint_{KKK}\left[\sum_{i=0}^{2m-3}\binom{2m-2}{i}(-1)^i\tilde{V}^i\right]dP_1\,dP_2\,dP_3\right)+$$

$$+\ldots+\beta_{2m-2k+2}\left(1-\frac{2}{2m-2k+3}-\frac{2(2m-2k+1)(2m-2k+2)}{12}\cdot\right.$$

$$\cdot\iiint_{KKK}\tilde{V}^{2m-2k}\,dP_1\,dP_2\,dP_3-\frac{(2m-2k+1)(2m-2k+2)}{12}\cdot$$

$$\left.\cdot\iiint_{KKK}\left[\sum_{i=0}^{2m-2k-1}\binom{2m-2k}{i}(-1)^i\tilde{V}^i\right]dP_1\,dP_2\,dP_3\right)+$$

$$+\ldots+\beta_4\left(1-\frac{2}{5}-2\iiint_{KKK}\tilde{V}^2\,dP_1\,dP_2\,dP_3-\iiint_{KKK}[-2\tilde{V}+1]\,dP_1\,dP_2\,dP_3\right). \tag{3.5}$$

Comparing the coefficients of $\iiint_{KKK}\tilde{V}^i\,dP_1\,dP_2\,dP_3$ $(i=0,1,\ldots,2m-2)$ in (3.5) we get

$$\frac{2m(2m+1)}{12}\binom{2m-1}{2m-2}=\frac{2(2m-1)2m}{12}\beta_{2m}, \tag{3.6}$$

$$\frac{2m(2m+1)}{12}\binom{2m-1}{2m-2k}=\frac{(2m-1)2m}{12}\binom{2m-2}{2m-2k}\beta_{2m}+$$

$$+\ldots+\frac{2(2m-2k+1)(2m-2k+2)}{12}\beta_{2m-2k+2}\quad\text{for } k=2,3,\ldots,m-1, \tag{3.7}$$

$$\frac{2m(2m+1)}{12}\binom{2m-1}{2m-2k-1} = \frac{(2m-1)2m}{12}\binom{2m-2}{2m-2k-1}\beta_{2m} + \\ + \ldots + \frac{(2m-2k+1)(2m-2k+2)}{12}\binom{2m-2k}{2m-2k-1}\beta_{2m-2k+2} \\ \text{for } k = 1, 2, \ldots, m-1 \qquad (3.8)$$

and

$$1 - \frac{2}{2m+2} - \frac{2m(2m+1)}{12} = \left(1 - \frac{2}{2m+1} - \frac{(2m-1)2m}{12}\right)\beta_{2m} + \\ + \ldots + \left(1 - \frac{2}{2m-2k+3} - \frac{(2m-2k+1)(2m-2k+2)}{12}\right)\beta_{2m-2k+2} + \\ + \ldots + \left(1 - \frac{2}{5} - 1\right)\beta_4 . \qquad (3.9)$$

From (3.6) and (3.7) we easily conclude that

$$\beta_{2m} = \frac{2m+1}{2} \qquad (3.10)$$

and

$$\beta_{2m-2k+2} = \frac{m(2m+1)}{(2m-2k+1)(2m-2k+2)} \cdot \\ \cdot \left[\binom{2m-1}{2m-2k} - \sum_{i=1}^{k-1} \frac{(2m-2i+1)(2m-2i+2)}{2m(2m+1)}\binom{2m-2i}{2m-2k}\beta_{2m-2i+2}\right] \\ \text{for } k = 2, 3, \ldots, m-1 , \qquad (3.11)$$

in accordance with (1.5′) and (1.5″).
Finally, expressions (3.8) and (3.9) can be verified by an adequate addition of the constants defined by (3.11).

4. Some remarks

Remark 1. For $m = 2$ expressions (1.4) and (1.5) become

$$V_4^{(2)}(K) = 2V_3^{(2)}(K) \quad \text{and} \quad V_5^{(3)}(K) = \tfrac{5}{2}V_4^{(3)}(K) ,$$

coincident with Buchta's formulas (1.2) and (1.3).

Remark 2. Some numerical values for the planar case:

$$V_4^{(2)}(K) = 2\,V_3^{(2)}(K)\,,$$
$$V_6^{(2)}(K) = 3\,V_5^{(2)}(K) - 5\,V_3^{(2)}(K)\,,$$
$$V_8^{(2)}(K) = 4\,V_7^{(2)}(K) - 14\,V_5^{(2)}(K) + 28\,V_3^{(2)}(K)\,,$$
$$V_{10}^{(2)}(K) = 5\,V_9^{(2)}(K) - 30\,V_7^{(2)}(K) + 126\,V_5^{(2)}(K) - 255\,V_3^{(2)}(K)\,.$$

Remark 3. Some numerical values for the three-dimensional case:

$$V_5^{(3)}(K) = \tfrac{5}{2}\,V_4^{(3)}(K)\,,$$
$$V_7^{(3)}(K) = \tfrac{7}{2}\,V_6^{(3)}(K) - \tfrac{35}{4}\,V_4^{(3)}(K)\,,$$
$$V_9^{(3)}(K) = \tfrac{9}{2}\,V_8^{(3)}(K) - 21\,V_6^{(3)}(K) + 63\,V_4^{(3)}(K)\,.$$

Remark 4. Recently, again Buchta [3], p. 96, generalizes his results (1.2) and (1.3) to higher dimensions. For an arbitrary d-dimensional convex body he shows that

$$V_{d+2}^{(d)}(K) = \frac{d+2}{2}\,V_{d+1}^{(d)}(K)\,.$$

Remark 5. Unfortunately, the procedure to relate the expected volume $V_n^{(d)}(K)$, $d = 2, 3$, of the convex hull of n random points to the integral expressions (2.3) and (3.2) cannot be extended to higher dimensions. Therefore it seems to be difficult to generalize our results to higher dimensions.

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