On some inequalities connected with Fermat's equation

Now, from ((10); (iii)) we have

$$\{\csc^2\omega\}^2 = \left\{\sum_{i=1}^{3} \csc^2\alpha_i\right\}^2 \leqq 3\sum_{i=1}^{3} \csc^4\alpha_i. \tag{15}$$

Thus the right-hand side of (14) is positive; it is zero if and only if $\alpha_1 = \alpha_2 = \alpha_3$. It follows that the right-hand in (9) is greater than or equal to $3/\omega$ with equality if and only if $\alpha_1 = \alpha_2 = \alpha_3$. This finishes the proof of Theorem 1.

We end this note by remarking that a straightforward application of Holder's inequality on (7) gives

$$\frac{3}{\omega^\lambda} \leqq \sum_{i=1}^{3} \frac{1}{(\alpha_i - \omega)^\lambda} \tag{16}$$

for every $\lambda \geqq 1$.

Faruk Abi-Khuzam
American University of Beirut, Lebanon

REFERENCES

1 Abi-Khuzam F.: Proof of Yff's Conjecture on the Brocard Angle of a Triangle. El. Math. *29*, 141–142 (1974).
2 Abi-Khuzam F.: Inequalities of Yff-type in the Triangle. El. Math. *35*, 80–81 (1980).
3 Abi-Khuzam F. and Boghossian A.: On Some Geometric Inequalities, to appear.
4 Johnson R: Advanced Euclidean Geometry. Dover Pbl., N.Y. (1960).
5 Mascioni V.: Zur Abschätzung des Brocardschen Winkels. El. Math. *41*, 98–101 (1986).

 0013-6018/88/030078-04$1.50+0.20/0

On some inequalities connected with Fermat's equation

1. Introduction

In 1856 I. A. Grünert ([3], see also [6] p. 226) proved that if n is an integer, $n \geqq 2$ and $0 < x < y < z$ are real numbers satisfying the equation

$$x^n + y^n = z^n \tag{1}$$

then

$$z - y < \frac{x}{n}. \tag{2}$$

This result was rediscovered by G. Towes [7], and then by D. Zeitlin [8].

In 1979 L. Meres [4] improved the result of Grünert, replacing (2) by

$$z - y < \frac{x}{a}, \quad \text{for } a = n + 1 - n^{2-n},\ n \geqq 2. \tag{3}$$

In [1] we improved the result of Meres, replacing (3) by

$$z - y < \frac{x}{n+1}, \quad \text{for } n \geqq 4. \tag{4}$$

Fell, Graz and Paasche [2] have proved that, if equation (1) has a solution in positive integers $x < y < z$, where $n \geqq 2$, then

$$x^2 > 2y + 1. \tag{5}$$

We mention also the result of Perisastri (1969): $z < x^2$ ([5]; [6] p. 226).
In this paper we establish the following theorems, which improve (4) and (5).

Theorem 1. *Let k be a positive integer. If*

$$n > [(2k+1)C_1], \quad C_1 = \frac{\log 2}{2(1 - \log 2)}$$

and if equation (1) has a solution in real numbers $0 < x < y < z$, then

$$z - y < \frac{x}{n+k}. \tag{6}$$

Theorem 2. *If n is an integer, $n \geqq 2$ and if equation (1) has a solution in real numbers $0 < x < y < z$, then*

$$z - y < \frac{x}{n} C(n), \quad \textit{where } C(n) = \log 2\left(1 + \frac{C_2}{n}\right), \quad \frac{\log 2}{2} < C_2 < \frac{\log 2}{\sqrt{2}}. \tag{7}$$

Theorem 3. *If equation (1) has a solution in positive integers $x < y < z$ for some $n > 2$, then*

$$x^2 > 2z + 1. \tag{8}$$

2. Proof of the Theorems

Proof of Theorem 1. If x, y, z are real numbers satisfying (1) for some positive integer n, and such that $0 < x < y < z$, write

$$x = \delta y \quad \text{with} \quad 0 < \delta < 1.$$

Hence by (1) we obtain

$$z = (\delta^n + 1)^{\frac{1}{n}} \cdot y$$

and

$$z - y = \frac{(\delta^n + 1)^{\frac{1}{n}} - 1}{\delta} \cdot x . \tag{9}$$

Since the function

$$t \mapsto \frac{(t^n + 1)^{\frac{1}{n}} - 1}{t}$$

is increasing for $0 < t < 1$, (9) implies

$$z - y < (2^{\frac{1}{n}} - 1) \cdot x . \tag{10}$$

For each $k > 0$ there is an $n_0(k)$ such that

$$(2^{\frac{1}{n}} - 1) < \frac{1}{n + k} \quad \text{for } n \geqq n_0(k) , \tag{11}$$

since

$$\lim_{n \to \infty} \left(1 + \frac{1}{n + k}\right)^n = e .$$

We now show that (11) holds with

$$n_0(k) = \frac{\log 2}{2(1 - \log 2)} \cdot (2k + 1) . \tag{12}$$

The inequality

$$2 < \left(1 + \frac{1}{n + k}\right)^n \tag{13}$$

is equivalent to

$$\log 2 < n \log\left(1 + \frac{1}{n + k}\right). \tag{14}$$

Since

$$\log\left(1+\frac{1}{n+k}\right)>\frac{2}{2(n+k)+1}\quad \text{for } (n+k)>0, \quad \text{(14) is true if}$$

$$\log 2<\frac{2n}{2(n+k)+1}.$$

Thus (11) is true if $n_0(k)$ is as in (12), and also if

$$n_0(k)=[(2k+1)C_1], \tag{15}$$

where $[u]$ denotes the integral part of u. The proof is complete. We have for example

$$n_0(1)=3, \quad n_0(2)=5, \quad n_0(3)=7, \ldots. \tag{16}$$

Proof of Theorem 2. From the proof of Theorem 1 it follows that

$$z-y<(2^{\frac{1}{n}}-1)\cdot x.$$

We have

$$2^{\frac{1}{n}}=1+\frac{\log 2}{n}+\frac{(\log 2)^2}{n^2\cdot 2!}\xi,$$

$$2^{\frac{1}{n}}-1=\frac{\log 2}{n}\left(1+\frac{\log 2}{2n}\xi\right), \quad \text{with } 1<\xi<2^{\frac{1}{n}}\leqq\sqrt{2}.$$

Thus

$$2^{\frac{1}{n}}-1=\frac{\log 2}{n}\left(1+\frac{C_2}{n}\right), \quad \text{where } \frac{\log 2}{2}<C_2<\frac{\log 2}{\sqrt{2}}. \tag{17}$$

From (10) and (17) we obtain

$$z-y<\frac{x}{n}\cdot C(n), \quad \text{where } C(n)=\log 2\left(1+\frac{C_2}{n}\right) \tag{18}$$

and

$$\frac{\log 2}{2}<C_2<\frac{\log 2}{\sqrt{2}}.$$

The proof is complete.

Proof of Theorem 3. We may assume that x, y, z are relatively prime. Indeed, if the theorem is true in this case, and if x, y, z are positive integers such that

$$x^n + y^n = z^n \text{ (some } n > 2) \quad \text{and} \quad (x, y, z) = d \quad \text{with } d > 1\,,$$

set $x = dx'$, $y = dy'$, $z = dz'$. Then $(x', y', z') = 1$, so that

$$(x')^2 \geqq 2z' + 1\,; \quad \text{on multiplying by } d \text{ we get}$$

$$2z + 1 < 2z + d = d(2z' + 1) \leqq d(x')^2 < x^2\,.$$

Now if $x < y < z$ are positive real numbers such that

$$x^2 + y^2 \leqq z^2\,,$$

then

$$x^n + y^n < z^n \quad \text{for } n > 2\,,$$

since

$$z^n \geqq z^{n-2}(x^2 + y^2) > x^{n-2} \cdot x^2 + y^{n-2} \cdot y^2 = x^n + y^n\,.$$

It follows that if equation (1) has a solution in positive integers $x < y < z$ for some $n > 2$, then

$$(*) \quad x^2 + y^2 > z^2\,.$$

Now if $z > y$ and y, z are integers, then $z \geqq y + 1$ and by (*),

$$x^2 > z^2 - y^2 \geqq z^2 - (z-1)^2 = 2z - 1\,,$$

whence

$$x^2 \geqq 2z\,.$$

Now $x^2 = 2z$ is impossible if $x^n + y^n = z^n$ and $(x, y, z) = 1$.
Therefore $x^2 > 2z + 1$.
The proof is complete.

Krystyna Białek
Department of Mathematics
Pedagogical University
Zielona Góra, Poland

REFERENCES

1 Białek K.: Remark on Fermat's equation. Discuss. Math. T. VII, 1985, 119–122.
2 Fell A. und Paasche I.: P 66 (Fermatproblem). Praxis der Math. *3*, 80 (1961).

3 Grünert I. A.: Wenn $n > 1$, so gibt es unter den ganzen Zahlen von 1 bis n nicht zwei Werte von x und y, für welche wenn z einen ganzen Wert bezeichnet, $x^n + y^n = z^n$ ist. Archiv. Math. Phys. *27*, 119–120 (1856).
4 Meres L.: About certain inequalities connected with the great theorem of Fermat. Zeszyt. Nauk. Polit. Śląskiej 560 Mat.-Fiz. *30*, 215–218 (1979).
5 Perisastri M.: On Fermat's last theorem. Amer. Math. Monthly *76*, 671–675 (1969).
6 Ribenboim P.: 13 lectures on Fermat's last theorem. Springer-Verlag, New York, Heidelberg, Berlin 1980.
7 Towes G.: On Fermat's last theorem. Amer. Math. Monthly *41*, 419–424 (1934).
8 Zeitlin D.: A note on Fermat's last theorem. Fibonacci Quart. *12*, 368–402 (1974).

 0013-6018/88/030078-06$1.50+0.20/0

Fastpythagoräische Quadrupel

1. In [1] beschreibt O. Frink Methoden, alle Lösungen der diophantischen Gleichungen $x^2 + y^2 = z^2 + 1$ und $x^2 + y^2 = z^2 - 1$ zu finden. Wir lösen mit anderen Methoden das analoge Problem für vier Variable. Ein Quadrupel (x, y, z, w) natürlicher Zahlen heisse *fastpythagoräisch* (FPQ), wenn es eine der beiden Gleichungen

$$x^2 + y^2 + z^2 = w^2 + 1 \tag{1}$$

oder

$$x^2 + y^2 + z^2 = w^2 - 1 \tag{2}$$

erfüllt. Gesucht ist ein Verfahren zur Bestimmung aller FPQ. Wir betrachten Lösungen, die nur durch Vertauschung von Komponenten auseinander hervorgehen, als nicht wesentlich verschieden.

2. Wir lösen zuerst (1). Rechnet man modulo 4, so erkennt man, dass x, y, z nicht alle gerade und nicht alle ungerade sein können.

Fall 1: x, y, w gerade, z ungerade. Der Ansatz $(x, y, z, w) = (2a, 2b, 2c - 1, 2c + 2r)$ führt auf

$$c = \frac{a^2 + b^2 - r^2}{2r + 1}.$$

Für jede Wahl von a, b, r mit den Nebenbedingungen $a \leq b$, $r \geq 0$, $r^2 < a^2 + b^2$ und $2r + 1 \mid a^2 + b^2 - r^2$ erhält man genau ein FPQ, wie man durch Einsetzen in (1) bestätigt. Für $r = 0$ ergibt sich so die Schar $(2a, 2b, 2a^2 + 2b^2 - 1, 2a^2 + 2b^2)$. $(2, 6, 5, 8)$ ist das FPQ mit kleinstem w, das nicht zu dieser Schar gehört. Ist $(a, b, r + 1)$ ein pythagoräisches Tripel, so entsteht das triviale FPQ $(2a, 2b, 1, 2r + 2)$. Weitere FPQ finden sich in Tabelle 1 am Schluss dieser Note.