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No	a	b	c	d	Number of primes given by $an^3 + bn^2 + cn + d$ for				
					$n < 100$	$n < 200$	$n < 300$	$n < 400$	$n < 500$
(1)	1	-220	16 119	-392 723	75	134	179	219	261
(2)	1	-199	13 190	-290 869	75	124	163	206	235
(3)	1	-160	8 547	-142 811	75	130	179	221	264
(4)	1	-159	8 420	-148 153	76	124	164	203	238
(5)	1	-151	7 610	-129 097	76	125	168	204	245
(6)	1	-150	7 493	-124 277	76	128	170	217	250
(7)	1	-137	6 270	-95 203	75	121	168	197	233
(8)	1	-125	5 196	-73 291	79	130	174	212	254
(9)	1	-119	4 718	-71 741	75	118	158	190	224
(10)	1	-114	4 343	-54 829	76	125	171	200	232
(11)	1	-111	4 100	-49 367	76	119	150	199	246
(12)	1	-97	3 126	-32 603	75	115	161	195	235
(13)	1	-96	3 059	-32 563	75	127	162	192	225
(14)	1	-82	2 237	-20 407	75	112	149	183	218
(15)	2	-489	39 847	-1 084 553	75	134	176	222	267
(16)	2	-372	23 050	-476 027	75	128	174	211	239
(17)	2	-292	14 202	-231 551	76	124	160	206	252
(18)	2	-289	13 917	-221 891	75	124	172	210	247
(19)	2	-281	13 157	-204 487	78	129	169	214	250
(20)	2	-280	13 072	-203 857	76	123	168	204	240
(21)	2	-199	6 595	-79 657	79	125	174	217	257

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A remark on the gamma function

According to Problem 188, Part II of G. Pólya and G. Szegő [2] for each integrable function $f(x)$ on $0 \leq x \leq 1$ we have:

$$\lim_{n \rightarrow \infty} \frac{1}{\varphi(n)} \sum_{\substack{k=1 \\ (k,n)=1}}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx,$$

or

$$\sum_{\substack{k=1 \\ (k,n)=1}}^n f\left(\frac{k}{n}\right) \sim \varphi(n) \int_0^1 f(x) dx \quad (n \rightarrow \infty),$$

where $\varphi(n)$ denotes Euler's totient function.

Let $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ be the Euler gamma function. Then using the above mentioned result for $f(x) = \log \Gamma(x)$ and taking into account that $\int_0^1 \log \Gamma(x) dx = \log \sqrt{2\pi}$ (Raabe's integral; [3]) we obtain

$$\sum_{\substack{k=1 \\ (k,n)=1}}^n \log \Gamma\left(\frac{k}{n}\right) \sim \varphi(n) \log \sqrt{2\pi},$$

or

$$\log P(n) \sim \varphi(n) \log \sqrt{2\pi}, \quad \text{where} \quad P(n) \equiv \prod_{\substack{k=1 \\ (k,n)=1}}^n \Gamma\left(\frac{k}{n}\right).$$

The aim of this note is to give an explicit formula for $P(n)$ and to establish an asymptotic formula with remainder term for the summatory function of $\log P(n)$.

We shall use the following results:

Lemma 1 ([3]).

$$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}, \quad n > 1.$$

Lemma 2. (A. Hurwitz, see Problem 35, Part VIII of [2])

Suppose $\psi(x)$ is an arbitrary function defined for $0 \leq x \leq 1$.

If $f(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^n \psi\left(\frac{k}{n}\right)$ and $g(n) = \sum_{k=1}^n \psi\left(\frac{k}{n}\right)$, then $f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$, μ denoting the

Möbius function.

Lemma 3.

If $F(n) = \prod_{\substack{k=1 \\ (k,n)=1}}^n \psi^*\left(\frac{k}{n}\right)$, $G(n) = \prod_{k=1}^n \psi^*\left(\frac{k}{n}\right)$ and $\psi^*(x) > 0$ for every x , $0 \leq x \leq 1$, then

$$F(n) = \sum_{d|n} [G(d)]^{\mu\left(\frac{n}{d}\right)}.$$

Proof: Apply Lemma 2 for $f = \log F$, $g = \log G$, $\psi = \log \psi^*$.

Theorem 1.

For $n > 1$ we have

$$P(n) = \frac{(2\pi)^{\varphi(n)/2}}{\exp \Lambda(n)/2} = \begin{cases} (2\pi)^{\varphi(n)/2} / \sqrt{p}, & \text{for } n = p^m \\ (2\pi)^{\varphi(n)/2}, & \text{for } n \neq p^m \end{cases}$$

where $\Lambda(n)$ is von Mangoldt's arithmetic function.

Proof: Using Lemma 1 und Lemma 3 one obtains successively:

$$P(n) = \prod_{d|n} \left[\frac{(2\pi)^{(d-1)/2}}{\sqrt{d}} \right]^{\mu\left(\frac{n}{d}\right)} = (2\pi)^{\frac{1}{2} \sum_{d|n} d\mu\left(\frac{n}{d}\right) - \frac{1}{2} \sum_{d|n} \mu\left(\frac{n}{d}\right)} / \sqrt{h(n)},$$

where

$$\sum_{d|n} d\mu\left(\frac{n}{d}\right) = \varphi(n); \quad \sum_{d|n} \mu\left(\frac{n}{d}\right) = 0 \quad (n > 1); \quad \text{and } h(n) \equiv \prod_{d|n} d^{\mu\left(\frac{n}{d}\right)}.$$

Here

$$\log h(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \log d = \Lambda(n),$$

cf. [1], and the proof is complete.

Lemma 4 (cf. [1]).

$$\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x),$$

$$\sum_{n \leq x} \Lambda(n) = O(x).$$

Theorem 2.

$$\sum_{n \leq x} \log P(n) = \frac{3 \log 2\pi}{2\pi^2} x^2 + O(x \log x).$$

Proof: By Theorem 1 and Lemma 4:

$$\begin{aligned} \sum_{n \leq x} \log P(n) &= \sum_{n \leq x} \left[\frac{\varphi(n)}{2} \log 2\pi - \frac{1}{2} \Lambda(n) \right] = \\ &= \frac{\log 2\pi}{2} \sum_{n \leq x} \varphi(n) - \frac{1}{2} \sum_{n \leq x} \Lambda(n) = \\ &= \frac{\log 2\pi}{2} \left(\frac{3}{\pi^2} x^2 + O(x \log x) \right) - \frac{1}{2} O(x) = \\ &= \frac{3 \log 2\pi}{2 \pi^2} x^2 + O(x \log x). \end{aligned}$$

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Kleine Mitteilungen

Zu einer Aufgabe der Kombinatorik

A_p^k ; $p \in \mathbb{N}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ sei die Anzahl der Möglichkeiten, k (nicht zu unterscheidende) Dinge auf p (zu unterscheidende) Personen aufzuteilen. Das Resultat

$$A_p^k = \binom{p+k-1}{k}; \quad p \in \mathbb{N}, k \in \mathbb{N}_0 \tag{1}$$

wird im allgemeinen durch vollständige Induktion gewonnen, wobei

$$A_p^2 = \binom{p+2-1}{2}, \quad A_p^3 = \binom{p+3-1}{3}$$