Zeitschrift: Elemente der Mathematik

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 44 (1989)

Heft: 3

Artikel: A remark on the gamma function

Autor: Sándor, J. / Harghita, Jud. / Tóth, L.

DOI: https://doi.org/10.5169/seals-41612

					Number of primes given by an^3+bn^2+cn+d for				
No	a	b	c	d	$n<100$	$n<200$	$n<300$	$n<400$	$n<500$
(1)	1	−220	16119	−392723	75	134	179	219	261
(2)	1	−199	13190	−290869	75	124	163	206	235
(3)	1	−160	8547	−142811	75	130	179	221	264
(4)	1	−159	8420	−148153	76	124	164	203	238
(5)	1	−151	7610	−129097	76	125	168	204	245
(6)	1	−150	7493	−124277	76	128	170	217	250
(7)	1	−137	6270	−95203	75	121	168	197	233
(8)	1	−125	5196	−73291	79	130	174	212	254
(9)	1	−119	4718	−71741	75	118	158	190	224
(10)	1	−114	4343	−54829	76	125	171	200	232
(11)	1	−111	4100	−49367	76	119	150	199	246
(12)	1	−97	3126	−32603	75	115	161	195	235
(13)	1	−96	3059	−32563	75	127	162	192	225
(14)	1	−82	2237	−20407	75	112	149	183	218
(15)	2	−489	39847	−1084553	75	134	176	222	267
(16)	2	−372	23050	−476027	75	128	174	211	239
(17)	2	−292	14202	−231551	76	124	160	206	252
(18)	2	−289	13917	−221891	75	124	172	210	247
(19)	2	−281	13157	−204487	78	129	169	214	250
(20)	2	−280	13072	−203857	76	123	168	204	240
(21)	2	−199	6595	−79657	79	125	174	217	257

P. Goetgheluck, Université de Paris-sud

REFERENCES

1 Dickson L. E.: History of the theory of numbers, vol. I. Carnegie Inst. Washington, Chelsea, New York 1952.
2 Guy R. K.: Reviews in number theory 1973–1983, vol. I. AMS, Providence RI, 1984.
3 Karst E.: New quadratic forms with high density of primes. Elem. Math. *28,* 116–118 (1973).
4 Leveque W. J.: Reviews in number theory 1940–1972, vol. I. AMS, Providence RI, 1974.

 0013-6018/89/030070-04$1.50+0.20/0

A remark on the gamma function

According to Problem 188, Part II of G. Pólya and G. Szegő [2] for each integrable function $f(x)$ on $0 \leqq x \leqq 1$ we have:

$$\lim_{n\to\infty} \frac{1}{\varphi(n)} \sum_{\substack{k=1 \\ (k,n)=1}}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx,$$

or

$$\sum_{\substack{k=1 \\ (k,n)=1}}^{n} f\left(\frac{k}{n}\right) \sim \varphi(n) \int_0^1 f(x)\,dx \quad (n \to \infty),$$

where $\varphi(n)$ denotes Euler's totient function.

Let $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt$ be the Euler gamma function. Then using the above mentioned result for $f(x) = \log \Gamma(x)$ and taking into account that $\int_0^1 \log \Gamma(x)\,dx = \log \sqrt{2\pi}$ (Raabe's integral; [3]) we obtain

$$\sum_{\substack{k=1 \\ (k,n)=1}}^{n} \log \Gamma\left(\frac{k}{n}\right) \sim \varphi(n) \log \sqrt{2\pi},$$

or

$$\log P(n) \sim \varphi(n) \log \sqrt{2\pi}, \quad \text{where} \quad P(n) \equiv \prod_{\substack{k=1 \\ (k,n)=1}}^{n} \Gamma\left(\frac{k}{n}\right).$$

The aim of this note is to give an explicit formula for $P(n)$ and to establish an asymptotic formula with remainder term for the summatory function of $\log P(n)$.

We shall use the following results:

Lemma 1 ([3]).

$$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}, \quad n > 1.$$

Lemma 2. (A. Hurwitz, see Problem 35, Part VIII of [2])

Suppose $\psi(x)$ is an arbitrary function defined for $0 \leqq x \leqq 1$.

If $f(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} \psi\left(\frac{k}{n}\right)$ *and* $g(n) = \sum_{k=1}^{n} \psi\left(\frac{k}{n}\right)$, *then* $f(n) = \sum_{d \mid n} \mu(d)\, g\left(\frac{n}{d}\right)$, *$\mu$ denoting the Möbius function.*

Lemma 3.

If $F(n) = \prod_{\substack{k=1 \\ (k,n)=1}}^{n} \psi^*\left(\frac{k}{n}\right)$, $G(n) = \prod_{k=1}^{n} \psi^*\left(\frac{k}{n}\right)$ *and* $\psi^*(x) > 0$ *for every* x, $0 \leqq x \leqq 1$, *then*

$$F(n) = \sum_{d \mid n} [G(d)]^{\mu\left(\frac{n}{d}\right)}.$$

Proof: Apply Lemma 2 for $f=\log F$, $g=\log G$, $\psi=\log\psi^*$.

Theorem 1.

For $n>1$ we have

$$P(n)=\frac{(2\pi)^{\varphi(n)/2}}{\exp\Lambda(n)/2}=\begin{cases}(2\pi)^{\varphi(n)/2}/\sqrt{p}, & \text{for} \quad n=p^m\\ (2\pi)^{\varphi(n)/2}, & \text{for} \quad n\neq p^m\end{cases},$$

where $\Lambda(n)$ is von Mangoldt's arithmetic function.

Proof: Using Lemma 1 und Lemma 3 one obtains successively:

$$P(n)=\prod_{d\mid n}\left[\frac{(2\pi)^{(d-1)/2}}{\sqrt{d}}\right]^{\mu\left(\frac{n}{d}\right)}=(2\pi)^{\frac{1}{2}\sum_{d\mid n}d\mu\left(\frac{n}{d}\right)-\frac{1}{2}\sum_{d\mid n}\mu\left(\frac{n}{d}\right)}\Big/\sqrt{h(n)},$$

where

$$\sum_{d\mid n}d\mu\left(\frac{n}{d}\right)=\varphi(n);\ \sum_{d\mid n}\mu\left(\frac{n}{d}\right)=0\ (n>1);\ \text{and}\ h(n)\equiv\prod_{d\mid n}d^{\mu\left(\frac{n}{d}\right)}.$$

Here

$$\log h(n)=\sum_{d\mid n}\mu\left(\frac{n}{d}\right)\log d=\Lambda(n),$$

cf. [1], and the proof is complete.

Lemma 4 (cf. [1]).

$$\sum_{n\leq x}\varphi(n)=\frac{3}{\pi^2}x^2+O(x\log x),$$

$$\sum_{n\leq x}\Lambda(n)=O(x).$$

Theorem 2.

$$\sum_{n\leq x}\log P(n)=\frac{3\log 2\pi}{2\pi^2}x^2+O(x\log x).$$

Proof: By Theorem 1 and Lemma 4:

$$\sum_{n \leqq x} \log P(n) = \sum_{n \leqq x} \left[\frac{\varphi(n)}{2} \log 2\pi - \tfrac{1}{2} \Lambda(n) \right] =$$

$$= \frac{\log 2\pi}{2} \sum_{n \leqq x} \varphi(n) - \tfrac{1}{2} \sum_{n \leqq x} \Lambda(n) =$$

$$= \frac{\log 2\pi}{2} \left(\frac{3}{\pi^2} x^2 + O(x \log x) \right) - \tfrac{1}{2} O(x) =$$

$$= \frac{3 \log 2\pi}{2\ \pi^2} x^2 + O(x \log x).$$

The authors wish to thank the referee for valuable suggestions.

J. Sándor, Jud. Harghita, Romania and L. Tóth, Satu Mare, Romania

REFERENCES

1 Hardy, G. H. and Wright, E. M.: An introduction to the theory of numbers. Clarendon Press, Oxford 1960.
2 Pólya, G. and Szegő, G.: Problems and theorems in analysis. Springer Verlag Berlin, Heidelberg 1972.
3 Whittaker, E. T. and Watson, G. N.: A course of modern analysis. Cambridge Univ. Press, 1969.

 0013-6018/89/030073-04\$1.50+0.20/0

Kleine Mitteilungen

Zu einer Aufgabe der Kombinatorik

A_p^k; $p \in \mathbb{N}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ sei die Anzahl der Möglichkeiten, k (nicht zu unterscheidende) Dinge auf p (zu unterscheidende) Personen aufzuteilen. Das Resultat

$$A_p^k = \binom{p+k-1}{k}; \quad p \in \mathbb{N}, k \in \mathbb{N}_0 \tag{1}$$

wird im allgemeinen durch vollständige Induktion gewonnen, wobei

$$A_p^2 = \binom{p+2-1}{2}, \quad A_p^3 = \binom{p+3-1}{3}$$