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1. In a paper published in 1836, C. H. Nagel [4] defines the "middlespoint" (Mittenpunkt) of a given triangle ABC in the following manner:

Let S_A, S_B, S_C be the midpoints of BC, CA, AB respectively and I_a, I_b, I_c the centres of the excircles, then the lines $S_A I_a$, $S_B I_b$, $S_C I_c$ concur at M, the middlespoint of ABC,

see also [1]. The name is probably derived from the fact that the point is constructed using "middles", namely, *centres* of circles and *midpoints* of line segments. In this paper, we derive a dual (line) for this remarkable, but seemingly little known, point and show how this new line relates to some known geometry of the triangle.

2. Desargues's two-triangle theorem in the plane states that if triangles ABC and $A_1B_1C_1$ are perspective from a point L, they are perspective from a line l, i.e. if $AA_1 \cap BB_1 \cap CC_1 = L$, then $(AB \cap A_1B_1) \cup (BC \cap B_1C_1) \cup (CA \cap C_1A_1) = l$. Clearly, the converse of this theorem is also its dual, hence, for purposes of this paper, we refer to L and l as "Desarguesian duals". Also, we will have occasion to make reference to the special case when the triangle $A_1B_1C_1$ is inscribed in ABC, i.e. $A_1$is on BC, etc. In this instance, L is called the trilinear pole of l and, dually, l is the trilinear polar of L, see [2].

3. In order to facilitate the arguments, we shall use a system of homogeneous coordinates called "trilinear" or "normal". In order to avoid a possible confusion with trilinear poles and polars, we shall use the term "normal" throughout. In this system, the coordinates (x, y, z) of a point L in the plane of ABC are proportional to the signed distances d_a, d_b, d_c of L from the sides of the triangle of reference ABC, where, obviously, $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$. The distance d_a, for example, is positive if L and the unit point $I = (1, 1, 1)$, the incentre, are on the same side of $a = BC$ and negative otherwise. For instance, the excentre I_A of the excircle opposite vertex A has coordinates $(-1, 1, 1)$, or its projective equivalent, $(1, -1, -1)$.
We now derive the normal line coordinates of l, the trilinear polar of L, that is, if a line t has equation $ux + vy + wz = 0$ then $t = [u, v, w]$ is its normal representation. Also, for the remainder of the paper, we shall denote A_1 by A_L, etc., thus $A_L = (0, y, z)$, $B_L = (x, 0, z)$, $C_L = (x, y, 0)$ and, consequently, $A_L B_L = [yz, xz, -xy] = \left[\frac{1}{x}, \frac{1}{y}, -\frac{1}{z}\right]$, $xyz \neq 0$. Now $C_L'^{[1*]} = AB \cap A_L B_L = (x, -y, 0)$ and, similarly $B_L' = (-x, 0, z)$, $A_L' = (0, y, -z)$; hence the coordinates of $l = A_L' B_L' C_L'$ readily follow. Since this result does not seem to appear in the available literature, we state it as a proposition.

Proposition 1. If a point $L = (x, y, z)$ is in the plane of, but not incident with, a given triangle of reference ABC then, l, its trilinear polar, has coordinates $\left[\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right]$.

4. The coordinates of the middlespoint M are also readily obtained. The coordinates of the centroid S are easily seen to be of the form $\left(\frac{1}{\sin\alpha}, \frac{1}{\sin\beta}, \frac{1}{\sin\gamma}\right)$, where α, β, γ are the measures of the vertex angles at A, B, C respectively, however, since $a = 2R \sin\alpha$, where R is the circumradius of ABC, it is more convenient to write $S = \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$. Now $I_a S_A = [(b-c), b, -c]$, $I_b S_B = [-a, (c-a), c]$, $I_c S_C = [a, -b, (a-b)]$, consequently, $M = (b+c-a, c+a-b, a+b-c)$. Again, we state this result in the form of a proposition.

Proposition 2. The normal coordinates of the middlespoint M of a given triangle ABC are given by $M = (s-a, s-b, s-c)$, where $s = \frac{a+b+c}{2}$.

5. Since the triangles $I_A I_B I_C$ and $S_A S_B S_C$ are perspective from the middlespoint M, they are, by Desargues's theorem, perspective from a line m, the Desarguesian dual of M, which we shall call the "middlesline" (mittenlinie) of ABC. Following the procedures above, it is an elementary exercise to show that the coordinates of m are $[a(s-a), b(s-b), c(s-c)]$, the details of which we leave as an exercise for the reader. We now state and prove a related result.

Proposition 3. The middlesline is the trilinear polar of the Gergonne point G of the given triangle ABC.
Proof: Since G_A, G_B, G_C are the points of contact of the incircle with the sides of ABC, $BG_A = s-b$ and $G_A C = s-c$, hence $G_A = (0, c(s-c), b(s-b))$ with similiar expressions for G_B, G_C. It now follows that the coordinates of G are $\left(\frac{1}{a(s-a)}, \frac{1}{b(s-b)}, \frac{1}{c(s-c)}\right)$ and by proposition 1, its trilinear polar is $[a(s-a), b(s-b), c(s-c)] = m$ as claimed.

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NOTE

[1*] The "prime" notation is particularly useful here since C_L and C'_L are related by the harmonic conjugacy involution, see [2].

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