A note on L'Hôpital's rule

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It now follows that $d(X)$ is not subharmonic in Ω for, if it were, then the subharmonic function $s(X) = d(X) + 2\varepsilon$ would satisfy property (iii a) giving

$$\int_D \{d(X) + 2\varepsilon\}\, dx_1\, dx_2 = \int_0^{\varepsilon} 2\pi M(s;(0,2\varepsilon),r)\, r\, dr \geq \pi\varepsilon^2 s(0,2\varepsilon) = 0\,.$$

The same paper gives a counterexample to show that Theorem 4 fails in higher dimensions. For example, when $n = 3$, let Ω be the torus obtained by rotating the disc

$$\{(0,x_2,x_3):(x_2-2)^2 + x_3^2 < 1\}$$

about the x_3-axis. Then it can be shown that u is subharmonic in Ω, yet Ω is clearly not convex. What can be said in higher dimensions is that, if we set $u(X) = \operatorname{dist}(X, \partial\Omega)$ for $X \in \boldsymbol{R}^n \backslash \Omega$, then the function u is subharmonic in the whole of $\boldsymbol{R}^n$ if and only if the domain Ω is a convex set (see [1] for details).

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A note on L'Hôpital's rule

1. Introduction

Recently the classical L'Hôpital's rule, $\lim f/g = \lim f'/g'$, has come again to the centre of interest. Referring to the basic article of Stolz [4], Boas [2] offered a general construction of counterexamples to the rule with non-monotonic g's. He pointed out that not the mere presence of zeros of g', but the infinite number of its sign changes may cause trouble with the rule. Clearly, by the intermediate value property of the derivative, g' can not change sign without having zeros. This is not the case for one-sided derivatives. Starting

from this observation, Vyborny-Nester [6] gave a version of L'Hôpital's rule using monotonicity theorems for one-sided derivatives.
The purpose of this paper is to find an exact condition showing that to what extent g may differ from being monotone for L'Hôpital still to hold. In particular, our results imply the known classical monotonic versions of the rule. In our considerations, we shall use the notion of absolute continuity and the Newton-Leibniz formula for Lebesgue integration. Thus throughout the paper, the expression «almost everywhere» (a.e.) and the integrals are to be taken according to the Lebesgue measure. We consider limits of functions at accumulation points of their domain and under the domain of a ratio f/g we mean the set of all those points x, where $f(x)$ and $g(x) \neq 0$ are defined.

2. General results

On the basis of Stolz [4], first we present a counterexample to L'Hôpital's rule for the case we intend to deal with.

Example 2.1. Let $\phi, \psi: [1, \infty[\to \boldsymbol{R}$ be defined by

$$\phi(\xi) = \xi + \sin\xi\cos\xi, \quad \psi(\xi) = e^{\sin\xi}\phi(\xi).$$

By Stolz [4], $\lim\limits_{\xi\to\infty}\phi(\xi) = \lim\limits_{\xi\to\infty}\psi(\xi) = +\infty$, $\lim\limits_{\xi\to\infty}\phi'(\xi)/\psi'(\xi) = 0$, however the ratio $\phi/\psi = e^{-\sin}$ has no limit at $+\infty$. Now for $f, g:]0,1] \to \boldsymbol{R}$ defined by

$$f(x) = 1/\phi(1/x), \quad g(x) = 1/\psi(1/x),$$

we have $\lim\limits_{x\searrow 0} f(x) = 0 = \lim\limits_{x\searrow 0} g(x)$ and also $\lim\limits_{x\searrow 0} f'(x)/g'(x) = \lim\limits_{x\searrow 0}[\phi'(1/x)/\psi'(1/x)]\,e^{2\sin(1/x)} = 0$, while the limit $\lim\limits_{x\searrow 0} f(x)/g(x) = \lim\limits_{x\searrow 0} e^{\sin(1/x)}$ does not exist.

Theorem 2.2. *Suppose that $f, g:]a,b[\to \boldsymbol{R}$ are absolutely continuous functions and $x_n \searrow a$ is a sequence in $]a,b[$. Under the assumptions*

i) $g'(x) = 0 \Rightarrow f'(x) = 0$, *a.e.* $x \in]a,b[$,

ii) $g(x_n) \neq 0, n \in \boldsymbol{N}$ *and* $\limsup\limits_{n\to\infty} \int_a^{x_n} |g'(t)|\,dt / |g(x_n)| = K < \infty$,

we have that if $\lim\limits_{x\searrow a} f(x) = 0 = \lim\limits_{x\searrow a} g(x)$ *and* $\lim\limits_{x\searrow a} f'(x)/g'(x) = L \in \boldsymbol{R}$, *then* $\lim\limits_{n\to\infty} f(x_n)/g(x_n) = L$.

Proof. We may and do assume that $L = 0$, since otherwise $f - Lg$ could be considered rather than f. Thus for every $\varepsilon > 0$, there is a $\delta > 0$ with $|f'(x)| < \varepsilon |g'(x)|$ for a.e. $x \in]a, a+\delta[$. Hence, for sufficiently large n's, we have $a < x_n < a + \delta$ and so

$$\left|\frac{f(x_n)}{g(x_n)}\right| = \left|\frac{1}{g(x_n)} \int_a^{x_n} f'(t)\,dt\right| \leq \frac{\varepsilon}{|g(x_n)|} \int_a^{x_n} |g'(t)|\,dt.$$

Therefore $\limsup_{n\to\infty} |f(x_n)/g(x_n)| \leqq \varepsilon K$.

Corollary 2.3. *Suppose that* $f, g\colon]a,b[\to \boldsymbol{R}$ *are absolutely continuous functions such that*

i) $g'(x) = 0 \Rightarrow f'(x) = 0$, *a.e.* $x \in]a,b[$,

ii) $\limsup_{x\searrow a} \int_a^x |g'(t)|\,\mathrm{d}t/|g(x)| = K < +\infty$.

If $\lim_{x\searrow a} f(x) = 0 = \lim_{x\searrow a} g(x)$ *and* $\lim_{x\searrow a} f'(x)/g'(x) = L \in \boldsymbol{R}$, *then* $\lim_{x\searrow a} f(x)/g(x) = L$.

3. Supplementaries

The general results above can easily be adapted for each version of L'Hôpital's rule. Moreover, [6], Theorem 1 is a special case of our Corollary 2.3, or more generally, for Dini derivatives, we have:

Theorem 3.1. *Let* $f, g\colon]a,b[\to \boldsymbol{R}$ *be continuous functions and suppose that (formulating, say, for the upper right-hand derivative)* $0 < D^+ g(x) < +\infty$ *for all but a countable many* $x \in]a,b[$. *Now whenever* $\lim_{x\searrow a} f(x) = 0 = \lim_{x\searrow a} g(x)$ *and* $\lim_{x\searrow a} D^+ f(x)/D^+ g(x) = L \in \boldsymbol{R}$, *then we have* $\lim_{x\searrow a} f(x)/g(x) = L$.

Proof. By [5], Corollary to Theorem 2, g is monotone increasing and so [3], Exercise (18.35) implies the absolute continuity of g on $]a,b[$. On the other hand, $|D^+ f| < (|L| + 1) D^+ g$ in a suitable neighbourhood $[a,c]$, except perhaps a countable subset. Hence, regarding that $D^+ g = g' = D_+ g$ a.e. in $[a,c]$, we have

$$0 < D^+ f + r D^+ g = D^+ f + r D_+ g \leqq D^+ (f + r g)$$

a.e. in $[a,c]$ with $r = |L| + 1 > 0$. Also, $D_+ g \geqq 0$ implies that

$$\begin{aligned} -\infty < -r D^+ g < D^+ f \leqq D^+ f + r D_+ g \leqq D^+ (f + r g) \\ \leqq D^+ f + r D^+ g < 2 r D^+ g < +\infty \end{aligned}$$

in $[a,c]$, apart from a countable number of exceptional points. This means, by repeating the above argument, that $f + r g$ is absolutely continuous on $[a,c]$, and so is f. Finally, since $g' > 0$ a.e., it follows that

$$\frac{1}{|g(x)|} \int_a^x |g'(t)|\,\mathrm{d}t = \frac{1}{|g(x)|} \left| \int_a^x g'(t) \right| = 1,$$

giving the property ii) of Corollary 2.3.

Remarks 3.2. a) We mention that there is also an elementary version of Corollary 2.3 above, in which the absolute continuity of f and g is replaced by the continuity of f' and g' on $]a,b[$. This version however, does not cover [6], Theorem 1 any more.

Notice that Corollary 2.3 is a proper generalization, as is shown by the function $g:]0,1[\to \boldsymbol{R}$, $g(x) = x + 2x^2 \sin(1/x)$.
b) Corollary 2.3 clearly shows that the essence of L'Hôpital's rule lies in the condition ii). Roughly speaking, this means that the variation of g in one direction must dominate that of in the other, a property replacing the monotonicity. Otherwise, there always appear certain «too small» values of $|g|$ comparing with $\int |g'|$ in any neighbourhood of $a+$, causing the failure of ii). Actually, condition ii) in Corollary 2.3 is essential for the validity of L'Hôpital's rule: If $\limsup_{x \searrow a} \int_a^x |g'(t)| \, dt / |g(x)| = +\infty$, then there always exists an absolutely continuous function $f:]a,b[\to \boldsymbol{R}$ such that $\lim_{x \searrow a} f'(x)/g'(x) = 0$ and simultaneously $\limsup_{x \searrow a} |f(x)/g(x)| = +\infty$. Indeed, one can choose easily a measurable function $m:]a,b[\to [0,1]$ such that $\lim_{x \searrow a} m(x) = 0$ and $\limsup_{x \searrow a} \int_a^x |g'(t)| m(t) \, dt / |g(x)| = +\infty$. Then f can be defined by

$$f(x) = \int_a^x |g'(t)| m(t) \, dt, \quad x \in]a,b[.$$

Example 3.3. As an illustration to our general results, consider on $]0,1[$ the functions

$$\bar{g}(t) = \begin{cases} 1 & \text{if } 2^{-n} < t < 3 \cdot 2^{-n-1}, \quad n = 1,3,5,\ldots \\ 0 & \text{elsewhere}, \end{cases}$$

$$\underline{g}(t) = \begin{cases} -1 & \text{if } 3 \cdot 2^{-n-1} < t < 2^{-n+1}, \quad n = 1,3,5,\ldots \\ 0 & \text{elsewhere}, \end{cases}$$

$$g_1(x) = \int_0^x 2\bar{g}(t) + \underline{g}(t) \, dt, \quad g_2(x) = \int_0^x (1+t)\bar{g}(t) + \underline{g}(t) \, dt.$$

The functions g_1 and g_2 are positive, while g_1 satisfies condition ii) but g_2 does not. All the same, their derivatives change sign according to the same rule.

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