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A generalization of Nagel's middlespoint

In an 1836 paper, C. H. von Nagel defines the «Mittenpunkt» of a given triangle Even though this point is readily constructed, it seems not to have found its way into the modern geometry of the triangle. In this paper, we show that by looking at the point from a slightly different point of view, one can obtain an infinite family of such points. In addition, we generalize a set of three related points.

1. Introduction

In what seems to be a little-known and somewhat inaccessible paper [4], C. H. von Nagel defines the *middlespoint* (Mittenpunkt) of a given triangle $A_1A_2A_3$ in the following manner.

Definition. *Let* S_i, $i = 1, 2, 3$, *denote the midpoints respectively of the sides* $A_{i+1}A_{i+2}$ *and* I^i, *the excentre opposite* A_i *of the triangle* $A_1A_2A_3$, *then* $\bigcap_i S_iI^i = M$, *the middlespoint of the given triangle.*

The name probably derives from the fact that the point is obtained using *middles*, i.e., *centres* of circles and *midpoints* of line segments. Even though this point has a simple construction using well-known concepts associated with the triangle, it seems not to appear in the available literature. One good recent paper on the subject known to us is by Baptist [1].
In this paper, we show that there exists an infinite family of such points each being a centre of perspectivity of a pair of triangles one circumscribed, the other inscribed, with respect to a given triangle. We also generalize a set of three related points referred to as *«interior middlespoint»* by Nagel. For the convenience of the reader, we supply some background.

2. Some properties of the middlespoint

In [5], Nagel also proves that the points M, the centroid S, and the Gergonne point $G = \bigcap_i A_iG_i$, where G_i is the contact point of the incircle with side $A_{i+1}A_{i+2}$ are collinear. We add a further result in the form of a theorem which we have not previously seen.

Theorem 1. *The middlespoint is collinear with the orthocentre H and E, the centre of the Spieker circle.*

Proof. We use *trilinear* or *normal* coordinates, for a point P in the plane of $A_1A_2A_3$. In this system, the coordinate triple (x_1, x_2, x_3) (often abbreviated to (x_i), $i = 1, 2, 3$) for P is such that x_i is proportional to the *signed* distances d_i of the given point from the side $A_{i+1}A_{i+2}$. The sign of d_i is positive or negative depending on whether or not P and the unit point I, the incentre, are in the same half-plane determined by I and $A_{i+1}A_{i+2}$. The relationship between the trilinear coordinate x_i and the actual distances d_i is given by

$$\frac{d_i}{x_i} = \frac{2\Delta}{\sum_{i=1}^{3} a_i x_i}$$

where Δ denotes the area of $A_1A_2A_3$ see [6]. Since the coordinates of I^i are $((-1)^{\delta_{ij}})$, $i, j = 1, 2, 3$, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, and those for S are $\left(\frac{1}{a_i}\right)$, where $a_i = \overline{A_{i+1}A_{i+2}}$, it is an elementary exercise to show that $M = (s - a_i)$, where $s = \frac{a_1 + a_2 + a_3}{2}$, the semiperimeter of $A_1A_2A_3$. The coordinates of H are $(\sec A_i)$ which, like S, may be found in standard sources, see for example [6]. Since we have not previously seen the coordinates for E, we provide some details. The Spieker circle is inscribed to the triangle $S_1S_2S_3$ and its centre E is the midpoint of IN, where N is the Nagel point, see [4]. The actual values of d_i for I and N are (r) and $\left(\frac{2r(s-a_i)}{a_i}\right)$ respectively and so the coordinates of E are $\left(\frac{a_{i+1} + a_{i+2}}{a_i}\right)$. The vanishing of the determinant

$$\begin{vmatrix} s - a_1 & s - a_2 & s - a_3 \\ \sec A_1 & \sec A_2 & \sec A_3 \\ \frac{a_2 + a_3}{a_1} & \frac{a_3 + a_1}{a_2} & \frac{a_1 + a_2}{a_3} \end{vmatrix}$$

was verified on an IBM pc using the *Reduce* compiler.

3. A generalization. We reconsider the definition of M and regard it from a different point of view. A point P in the interior of the triangle $A_1A_2A_3$ determines an inscribed triangle $P_1P_2P_3$, where $P_i = A_iP \cap A_{i+1}A_{i+2}$. A second interior point Q determines a triangle, $Q^1Q^2Q^3$ circumscribed about $A_1A_2A_3$, in the following manner. Consider the harmonic conjugates Q'_1, Q'_2, Q'_3 of Q_1, Q_2, Q_3 with respect to A_2 and A_3, A_3 and A_1, A_1 and A_2 which lie on a line q, the trilinear polar of Q with respect to the triangle $A_1A_2A_3$, see [2]. The point Q^i is the intersection of the lines $A_{i+1}Q'_{i+1}$ and $A_{i+2}Q'_{i+2}$. If Q has trilinear coordinates (y_i), then it is readily seen that those for Q^i are $[(-1)^{\delta_{ij}} y_j]$. We may now state and prove the following theorem.

Theorem 2. *The triangles $P_1P_2P_3$ and $Q^1Q^2Q^3$ are perspective from the point $T = P_iQ^i$, $i = 1, 2, 3$.*

Proof. We begin by stating the Ceva criterion for the concurrency of the lines A_iX_i with respect to the triangle $A_1A_2A_3$, where $X_i \in A_{i+1}A_{i+2}$. The lines A_iX_i are concurrent at a point X if and only if

$$\prod \frac{\overline{X_iA_{i+1}}}{\overline{X_iA_{i+2}}} = -1; \quad i = 1, 2, 3,$$

and the segments are *directed* in the sense that, for example, $\overline{XY} + \overline{YX} = 0$. This is the *line segment* criterion. Also, we have the equivalent *angle* criterion, i.e., $\bigcap_i A_iX_i = X$ if and only if

$$\prod \frac{\sin(< X_iA_iA_{i+1})}{\sin(< X_iA_iA_{i+2})} = -1,$$

with angles measured in a clockwise sense taken as negative, see [4]. We proceed to show that the *line segment* criteria for the points P and Q induce, in a natural way, the *angle* criterion on the lines P_iQ^i.

From figure 1,

$$\prod \frac{\overline{P_iA_{i+1}}}{\overline{P_iA_{i+2}}} = -\prod \frac{\overline{A_{i+1}Q^i}}{\overline{A_{i+2}Q^i}} \cdot \frac{\sin < A_{i+1}Q^iP_i}{\sin < A_{i+2}Q^iP_i},$$

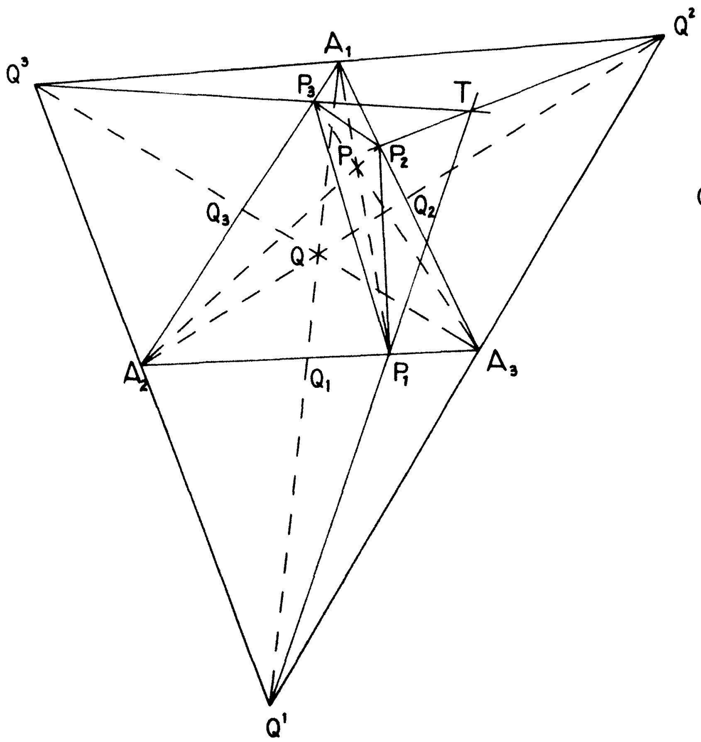

Figure 1. The generalized middlespoint T.

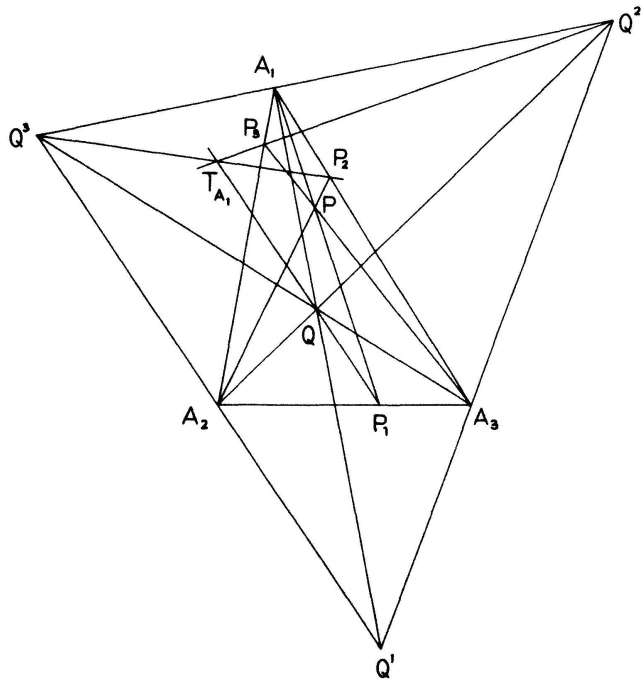

Figure 2. The generalized interior middlespoint T_{A_1}.

where also the angles are directed, i.e., $\sin(\angle A_{i+1} P_i Q^i) + \sin(\angle A_{i+2} P_i Q^i) = 0$. But, since the triangles $A_1 A_2 A_3$ and $Q^1 Q^2 Q^3$ are perspective from Q,

$$\prod \frac{A_{i+1} Q^i}{A_{i+2} Q^i} = -1,$$

hence

$$\prod \frac{\sin(\angle A_{i+1} Q^i P_i)}{\sin(\angle A_{i+2} Q^i P_i)} = -1$$

and $\bigcap_i P_i Q^1 = T$ as claimed.

If the coordinates of P are (z_i), then $P_i = ((1 - \delta_{ij}) z_j)$ and so the coordinates of T are $\left(y_i \sum_{j=1}^{3} \frac{(-1)^{\delta_{ij}} y_j}{z_j}\right)$. We leave the details of this derivation as an exercise for the reader. In addition to M, a second interesting special case of T is found by again taking Q as the incentre and P as G, the Gergonne point of $A_1 A_2 A_3$, see [3]. Since, for example, $A_2 G_1 = s - a_2$, the coordinates of G are $\left(\frac{1}{a_i(s - a_i)}\right)$ and those for T are $\left(\frac{1}{s - a_i}\right)$, which are the reciprocals of those for M. This latter point is called the *isogonal conjugate* of M, see [6], and is obtained geometrically by reflecting each of lines $A_i M_i$ in the corresponding internal angle bisector through A_i.

4. Additional points of a related type

Nagel further shows that by replacing one excentre by the incentre and interchanging the other two, one obtains three points of the form

$$M_{A_i} = S_i I \cap S_{i+1} I_{i+2} \cap S_{i+2} I_{i+1},$$

which Nagel calls the *interior* middlespoints of the given triangle. It is an elementary exercise to show that the trilinear coordinates of M_{A_1}, for example, are $(s, s - c, s - b)$ which since all three components are positive implies that M_{A_1} is in the interior of $A_1 A_2 A_3$.

Each of these points may be generalized in a similar fashion as the middlespoint.

Theorem 3. *If, in the definition of T, we replace Q_i by Q and interchange Q_{i+1} and Q_{i+2} we obtain three points of the form*

$$T_{A_i} = P_i Q \cap P_{i+1} Q^{i+2} \cap P_{i+2} Q^{i+1}; \quad i = 1, 2, 3.$$

Proof. If we denote by Γ the product of the sines of the angles at Q, Q^2, Q^3 determined by the lines QP_1, $Q^2 P_3$, $Q^3 P_2$ then

$$\prod \frac{\overline{P_i A_{i+1}}}{\overline{P_i A_{i+2}}} = -\left(\frac{\overline{A_2 Q}}{\overline{A_3 Q}} \cdot \frac{\overline{A_3 Q^3}}{\overline{A_1 Q^3}} \cdot \frac{\overline{A_1 Q^2}}{\overline{A_2 Q^2}}\right) \cdot \Gamma.$$

By inspecting figure 2 and considering the relationships

$$\frac{\overline{A_2 Q}}{\overline{A_3 Q}} = \frac{\overline{A_2 Q^3}\sin(< A_3 Q^3 A_2)}{\overline{A_3 Q^2}\sin(< A_3 Q^2 A_2)}, \quad \frac{\overline{A_3 Q^3}}{\overline{A_2 Q^2}} = \frac{\sin(< Q^3 A_2 A_3)}{\sin(< A_3 Q^3 A_2)} \cdot \frac{\sin(< A_3 Q^2 A_2)}{\sin(< Q^2 A_3 A_2)},$$

$$\frac{\overline{A_3 Q^1}}{\overline{A_2 Q^1}} = \frac{\sin(< A_3 A_2 Q^3)}{\sin(< A_2 A_3 Q^2)},$$

we see that the bracketed product is equal to

$$\prod \frac{\overline{A_i Q^{i+1}}}{\overline{A_i Q^{i+2}}} = -1,$$

hence $\Gamma = -1$ and the lines $P_1 Q$, $P_2 Q_3$, $P_3 Q_2$ are concurrent at T_{A_1}. Similar arguments hold for T_{A_2} and T_{A_3}.
The coordinates for T_{A_i} may be obtained directly or from those for T by noting the relevant differences in the construction of the required point. For example, the coordinates of T_{A_1} are given by

$$x_1 = y_1\left(\frac{y_1}{z_1} + \frac{y_2}{z_2} + \frac{y_3}{z_3}\right),$$

$$x_2 = y_2\left(\frac{y_1}{z_1} + \frac{y_2}{z_2} - \frac{y_3}{z_3}\right),$$

$$x_3 = y_3\left(\frac{y_1}{z_1} - \frac{y_2}{z_2} + \frac{y_3}{z_3}\right).$$

As an example, we see that the coordinates for the interior middlespoint M_{A_1} are as given previously with similar representations for M_{A_2} and M_{A_3}. In addition, it is easy to see that T_{A_i} is not necessarily in the interior of the triangle $A_1 A_2 A_3$.

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