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Beispiel 2. Komplexes Polynom mit einer 3-fachen Nullstelle:

$$P(z) = (z - (1 + i))^3 = z^3 - (3 + 3i)z^2 + (6i)z + (2 - 2i)$$

Simultane Iteration: (Fehlergrenze 0,0001)

Startwerte

```
0.95 + 0.98i      0.85 + 1.1i      1 + [illegible]i
```

```
Die iterierten Losungen sind:
```

[illegible] + 0.[illegible]i	0.9[illegible] + 1.1[illegible]i	1.0[illegible] + [illegible]i
0.959 + 0.981i	0.985 + 1.074i	1.04 + [illegible]i
0.970 + 0.98[illegible]i	0.995 + 1.038i	[illegible]
0.981 + 0.988i	0.998 + 1.021i	1.01[illegible] + [illegible]i
0.989 + 0.992i	0.999 + 1.012i	1.00[illegible] + [illegible]i
0.994 + 0.995i	0.999 + 1.007i	1.00[illegible] + [illegible]i
0.997 + 0.997i	0.999 + 1.004i	1.00[illegible] + [illegible]i
0.998 + 0.998i	0.999 + 1.002i	1.00[illegible] + 1.00[illegible]i
0.999 + 0.999i	1.000 + 1.001i	1.00[illegible] + 1.00[illegible]i
0.999 + 0.999i	1.000 + 1.001i	1.001 + [illegible]i
1.000 + 1.000i	1.000 + 1.000i	1.000 + 1.000i
1.000 + 1.000i	1.000 + 1.000i	1.000 + 1.000i
1.000 + 1.000i	1.000 + 1.000i	1.000 + 1.000i
1.000 + 1.000i	1.000 + 1.000i	1.000 + 1.000i

An dieser Stelle danke ich Frau C. Bandle, Basel für die Beratung bei der Abfassung dieser Arbeit.

R. Wyss, Kantonsschule Solothurn

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Moessnerian theorems. How to prove them by simple graph theoretical inspection

Introduction

Alfred Moessner formulated his original theorem about 40 years ago. It states the following:

Theorem (Moessner, 1951). Let k be a positive integer.
Perform the following algorithm:

STEP 1: Write down the sequence of integers $(n)_{n=1}^{\infty}$

STEP $t\,(2 \leqq t \leqq k)$: Leave out every $(k-2+t)^{\text{th}}$ term of the preceding sequence and write down the sequence of partial sums of the remaining sequence.

Then, at step k, we end up with the sequence of k^{th} powers $(i^k)_{i=1}^{\infty}$.

As an example, we take $k = 3$:

1	2	3	4	5	6	7	8	9	10	11	12	13	......
1	3		7	12		19	27		37	48		61	......
1			8			27			64			125	......

So we arrive at the sequence of cubes
From 1951 until 1966 several proofs and generalisations were formulated, e.g. by Perron, Paasche, Salié, van Ijzeren, Kazandzidis and Long (cf. Leveque [3]), but during the last two decades, as far as the author knows (cf. Leveque [3], Guy [1]) no contributions have been written except two papers in the Mathematical Gazette (Long [4], Slater [7]), and one in the Fibonacci Quarterly (Long [5]), that only give illustrative examples of Moessnerian theorems, and that refer to proofs given by Paasche ([6]), which they consider to be «somewhat involved and not really suitable at school level».
Besides, all proofs that the author knows from the literatur quoted by Leveque ([3]) are rather complicated by their extensive use of binomial identities, generating functions, or linear transformation techniques. In this paper we shall discuss a surprisingly simple graph theoretical method to prove both Moessner's original theorem and its generalisations. As a matter of fact the author is convinced of the possibilities to explain this method easily at school level. It is even a challenge for the readers to find graph theoretical proofs for general situations, as soon as they have seen the proof of the original theorem.

Graph theoretical proof of Moessner's theorem

Our basic graph theoretical argument is the following:
Suppose *every* edge in a graph G is *directed*, and A and B are any two vertices of G. Let H be the graph that originates from G by *reversing* the directions of *all* of its edges. Then the number of paths in G from A to B is equal to the number of paths in H from B to A.
The proof of this fact is obvious: Take every path in its reversed direction.
The question is how to transform Moessner's process in a graph theoretical setting. Therefore we first insert an extra STEP 0 in Moessner's algorithm so that STEP 1 also fits into the inductive scheme of the other steps. This is an easy enterprise: We take

STEP 0: Write down the all-one sequence $(1)_{n=1}^{\infty}$.

Our example $k = 3$ then looks as follows:

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	......
1	2	3		4	5	6		7	8	9		10	11	12		13	......
1	3			7	12			19	27			37	48			61	......
1				8				27				64				125	......

The graph G_3 corresponding to this example looks like the street map of an American city:
Situate a vertex at the position of every entry in Moessner's extended diagram, and draw an edge between every neighbouring pair in every row and column. Let every horizontal edge be directed to the East, and every vertical edge to the South (see Figure 1).

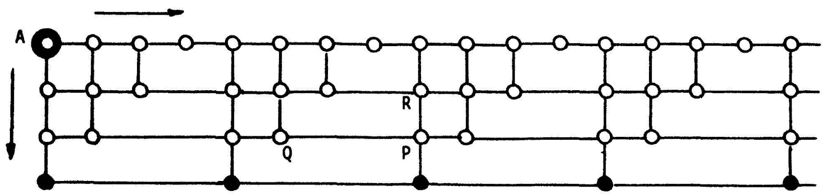

Figure 1. The directed graph G_3.

We situate the vertex A in the North West corner of the graph, and observe that any other vertex P can be reached from its western neighbour Q and its northern neighbour R, so the number of paths getting from A to P is equal to the sum of the numbers of paths leading to Q and R.
But this is exactly the way of forming partial sums. And, since the entries at the leftmost and topline of the Moessner diagram are all ones, all Moessner's entries in the extended diagram are the corresponding path counting numbers!
So let us have a look at the reversed graph H_3 in order to count paths leading from vertex B_j up to vertex$A$ (see Figure 2).

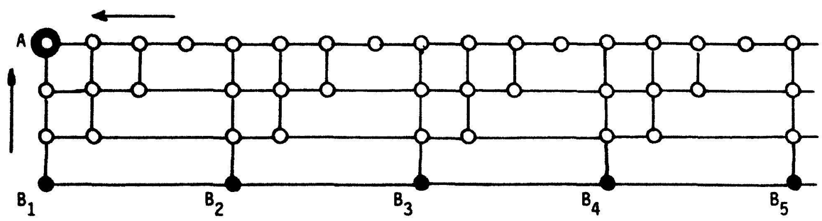

Figure 2. The directed graph H_3.

Path counting starting at B_j is an easy job when we perform it triangle-by-triangle by induction, and gives rise to simple geometric sequences:

	Triangle T_{j-3}					Triangle T_{j-2}					Triangle T_{j-1}					
--	64	48	36	27	--	27	18	12	8	--	8	4	2	1	--	1
--	16	12	9	.	--	9	6	4	.	--	4	2	1	.	--	1
--	4	3	.	.	--	3	2	.	.	--	2	1	.	.	--	1
--	1	.	.	.	--	1	.	.	.	--	1	.	.	.	--	1
	B_{j-3}					B_{j-2}					B_{j-1}					B_j

Figure 3. Path counting in H_3, starting at B_j.

The general case can be described in a huge graph with a multitriangular structure (see Figure 4, which shows the reversed version).

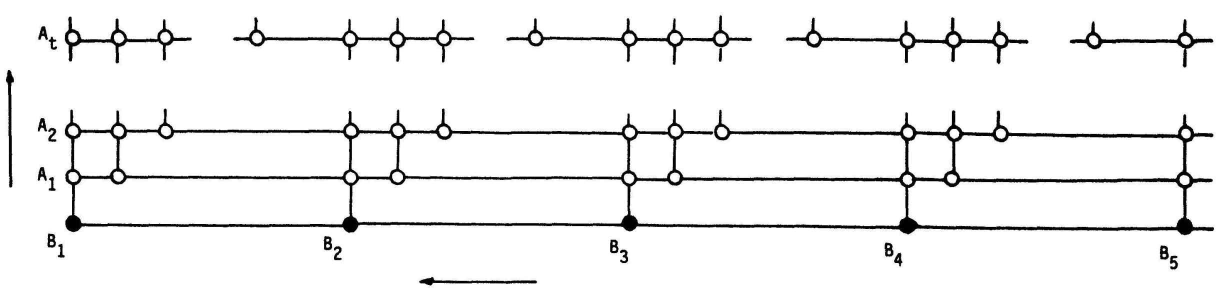

Figure 4. The multitriangular (reversed) graph H.

The inductive path counting step is illustrated in Figure 5, where $b := a + 1$.

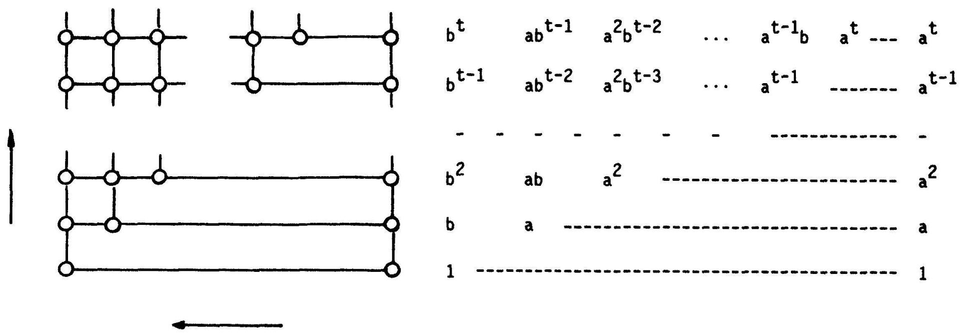

Figure 5. Path counting by induction in H up to level $t(b := a + 1)$.

Generalisations of Moessner's theorem

Generalisations of Moessner's problem can be obtained in the following way:
Let $(k_i)_{i=1}^{\infty}$ be a *non-decreasing* sequence of positive integers, and let $(p_n)_{n=1}^{\infty}$ be a sequence of real (complex) numbers. Consider the multitriangular array of positions depicted in Figure 6. Each triangle T_i has size k_i $(i = 1, 2, 3, \ldots)$.

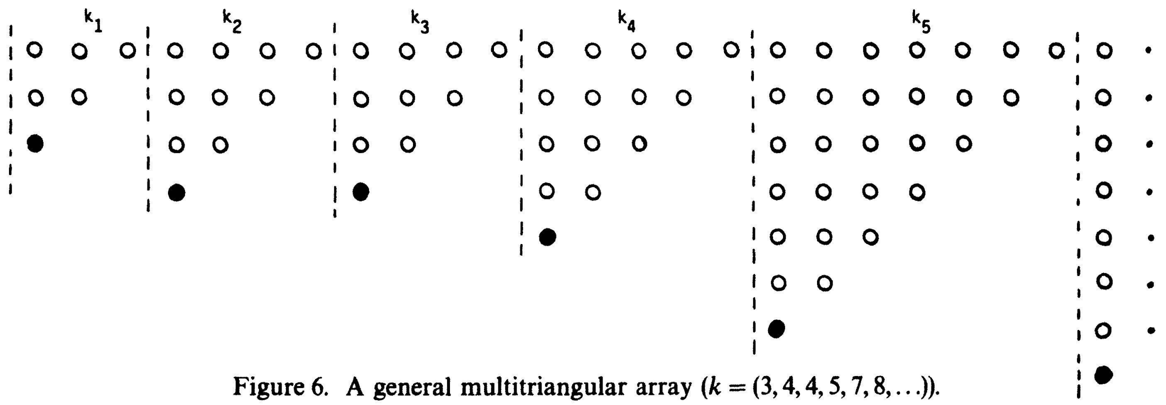

Figure 6. A general multitriangular array ($k = (3, 4, 4, 5, 7, 8, \ldots)$).

Write down the sequence $(p_n)_{n=1}^{\infty}$ on the first line of this array. Skip the last element of this sequence in each of the triangles, and write down the sequence of partial sums of the remaining elements on the second line in the corresponding positions. Continue this process. Question: What can be said about the sequence $(c_i)_{i=1}^{\infty}$ of ultimately skipped numbers q_1 in triangle T_i $(i = 1, 2, 3, \ldots)$?

Moessner's original problem is obtained by taking $k_i = k$, $p_n = n$ (or, equivalently, $k_i = k + 1$, $p_n = 1$).
Further examples are

$k_i = k$, p_n arbitrary (Salié, Paasche)
k_i arbitrary, $p_n = 1$ (Paasche)
$k_i = k$, $p_n = an + b$ (Long)
k_i arbitrary, $p_n = n$ (Kazandzidis).

We shall indicate the solution of the general problem by graph theoretical methods in two steps:

A. We take (k_i) arbitrary, $p_n = 1$.
B. We make use of the linearity of the mapping $(p_n)_{n=1}^{\infty} \mapsto (q_i)_{i=1}^{\infty}$, when we consider a fixed sequence $(k_i)_{i=1}$.

A. The graph theoretical interpretation of the all-one case for arbitrary non-decreasing k_i is a straightforward extension of the method in the preceding Section: We have to count paths from the bottom vertices in the (reversed) Moessner-graph of Figure 7, which in this particular case is chosen to be graph corresponding to Figure 6.

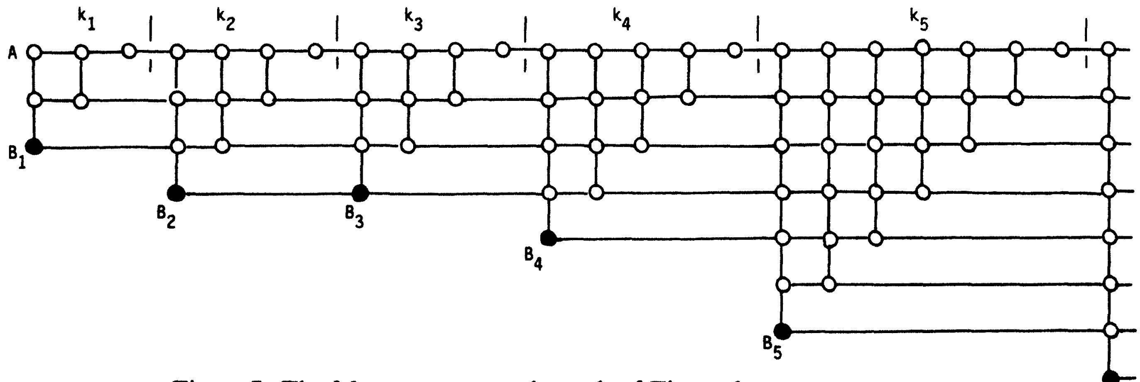

Figure 7. The Moessner reversed graph of Figure 6.

The general path counting number in this graph is again easily calculated triangle-by-triangle from right to left.
We find for its value q_i in the left upper vertex, when we start in the bottom vertex of triangle T_i:

$$q_i = 1^{k_i - k_{i-1}} * 2^{k_{i-1} - k_{i-2}} * \ldots * (i-1)^{k_2 - k_1} * i^{k_1 - 1}.$$

where $*$ denotes multiplication.
This result was found, more or less explicitly, by Paasche and by Kazandzidis.
At this point it is useful to mention that also the sequence $(e)^{(v)}$ consisting of $v - 1$ consecutive zeroes followed by the all-one sequence can be treated for arbitrary k_i, since it behaves as the all-one sequence for a slightly modified (also non-decreasing!) sequence (k_i').
B. It is evident that for an arbitrary starting sequence $(p_n)_{n=1}$ we can write

$$(p_n) = p_1 (e)^{(1)} + \sum_{v=2}^{\infty} (p_v - p_{v-1})(e)^{(v)}.$$

The linearity of the Moessner-mapping $(p_n) \mapsto (q_i)$ for fixed (k_i) finishes the job. There are no convergence problems in the infinite summation formula, because every q_i is affected only by a finite number of terms in this summation.
As an example, we take $p_n = n$ and $k_i = i$:
Figure 8 shows that we end up with the factorials!

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	2		6	11		18	26	35		46	58	71	85		101
			6			24	50			96	154	225			326
						24				120	274				600
										120					720
															720

Figure 8. The factorials are obtainable in the Moessner-diagram in an obvious way!

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Kleine Mitteilungen

Zur linearen Unabhängigkeit von Quadratwurzeln über den rationalen Zahlen

Wir nennen eine Quadratwurzel $\sqrt{a}$ «reduziert», wenn a eine quadratfreie ganz-rationale Zahl ist. Die nichttrivialen reduzierten Quadratwurzeln sind umkehrbar eindeutig den quadratischen Erweiterungskörpern des rationalen Zahlkörpers $\mathbb{Q}$, die «triviale» Wurzel $\sqrt{1}$ dem Körper $\mathbb{Q}$ selbst zugeordnet.
Wir werden zeigen, daß die Menge der reduzierten Quadratwurzeln, d.h. jede endliche Teilmenge, linear unabhängig über $\mathbb{Q}$ ist.
Dieses Ergebnis ist – in wesentlich allgemeinerer Form – bekannt [1*]. Bei dieser Note geht es daher um die Methode der Herleitung, die eine, wie ich meine, hübsche und nicht