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In this article we introduce an interesting transformation (called $GT_l(k)$-transformation (duality), cf. [2]) for any triangle and discuss its applications in producing new geometrical inequalities from well-known ones. Examples will be given in which the $GT_l(k)$-dual inequality is better than the initial inequality.

In a triangle ABC we denote by a, b, c the lengths of the sides BC, AC and AB respectively and by F, R, r its area and the radii of its circumscribed and inscribed circle. As usual, the letter s denotes the semiperimeter $\frac{1}{2}(a+b+c)$ of the triangle. Also if f is a function of three variables, $\Sigma f(a,b,c)$ (resp. $\Pi f(a,b,c)$) denotes the sum (resp. the product), of the values of $f(a,b,c)$ over all cyclic permutations of a, b, c.

Theorem. *Let ABC be triangle with the usual elements a, b, c, F, R, r. Then there also exists a triangle $A'B'C'$ with elements*

$$\begin{aligned} a' &= [a(2ks-(k+1)a)]^{1/2} \\ b' &= [b(2ks-(k+1)b)]^{1/2} \\ c' &= [c(2ks-(k+1)b)]^{1/2} \\ F' &= F(2k(k-1)R/r+1)^{1/2}, \\ R' &= (R/2)^{1/2}\ [k(k-1)^2s^2+(k+1)^2r(2(k-1)R+kr)]^{1/2}[2k(k-1)R+r]^{-1/2}, \end{aligned} \tag{1}$$

where k is any real number greater than or equal to one.

Proof. It is well-known that from the existance of a triangle with sides a_1, b_1, c_1 follows the existence of a triangle with sides $a_1^{1/2}$, $b_1^{1/2}$, $c_1^{1/2}$. Let $a_1=a'^2$, $b_1=b'^2$, $c_1=c'^2$, where a', b', c' are given by (1). Such a triangle exists because $a_1+b_1>c_1$, i.e. $a'^2+b'^2>c'^2$ implies

$$a[2ks-(k+1)a]+b[2ks-(k+1)b]>c[2ks-(k+1)c],$$

which is equivalent to

$$\begin{aligned} &2ks(a+b-c)>(k+1)(a^2+b^2-c^2), && \text{i.e.} \\ &k[(a+b)^2-c^2)>(k+1)(a^2+b^2-c^2), && \text{i.e.} \\ &2kab>a^2+b^2-c^2=2ab\cos C, && \text{i.e.} \\ &k>\cos C \end{aligned} \tag{2}$$

and conversely. The inequality (2) is obvious because $k\geqq 1$. Thus, there exists a triangle with sides $a_1^{1/2}=a'$, $b_1^{1/2}=b'$, $c_1^{1/2}=c'$.

Further, with the help of the well-known equalities: $\Sigma bc=s^2+r^2+4Rr$, $\Sigma a^2=2(s^2-r^2-4Rr)$, $\Sigma a^3=2s(s^2-3r^2-6Rr)$, $\Sigma bc(b+c)=2s(s^2+r^2-2Rr)$ it is not difficult to compute the area F' and the circumradius R' of the triangle $A'B'C'$:

$$\begin{aligned}16F'^2 &= 2\Sigma b'^2 c'^2 - \Sigma a'^4\\ &= 2\Sigma b\,c\,[2ks-(k+1)b]\,[2ks-(k+1)c] - \Sigma a^2[2ks-(k+1)a]^2\\ &= 16F^2[2k(k-1)R/r+1],\end{aligned}$$

i.e. $F' = F[2k(k-1)R/r+1]^{1/2}$,

$$\begin{aligned}R' &= [abc\,\Pi\,[2ks-(k+1)a]]^{1/2}/4F[2k(k-1)R/r+1]^{1/2}\\ &= (R/2)^{1/2}[k(k-1)^2s^2+(k+1)^2r(2(k-1)R+kr)]^{1/2}[2k(k-1)R+r]^{-1/2}.\end{aligned}$$

The proof is complete.
Consequently, for any triangle inequality (or equality) of the form

$$I(a,b,c,F,R) \gtreqless 0 \tag{3}$$

we also have the dual inequality (or equality)

$$I_{GT_I(k)} \equiv I(a',b',c',F',R') \gtreqless 0 \tag{4}$$

where a', b', c', F', R' one given by (1), i.e.

$$I \gtreqless 0 \xrightarrow{GT_I(k)} I_{GT_I(k)} \gtreqless 0,$$

where $k \geqq 1$ is any real number. (In [2] the $GT_I(1)$-transformation is also called T_I-transformation).
We will illustrate the applications of $GT_I(k)$-duality with the following characteristic examples (GI $\alpha.\beta$ denotes the geometric inequality in [1] with numbering $\alpha.\beta$).

Example 1.

$$\text{GI 4.4: } \Sigma a^2 \geqq 4F\sqrt{3} \xrightarrow{GT_I(k)} 2k\Sigma bc - \Sigma a^2 \geqq 4\sqrt{3}F(2k(k-1)R/r+1)^{1/2}. \tag{5}$$

Remark 1.1. From (5) for $k = 1$ we obtain the Finsler-Hadwiger inequality GI 4.7 – the left one, which is sharper than GI 4.4.

Remark 1.2. From GI 5.1 of [1] and (5) follows the inequality

$$2k\Sigma bc - \Sigma a^2 \geqq 4\sqrt{3}F(2k(k-1)R/r+1)^{1/2} \geqq (2k-1)4\sqrt{3}F. \tag{5'}$$

Example 2. GI 4.12: $\Sigma b^2c^2 \geqq 16F^2 \xrightarrow{GT_I(k)}$

$$\Sigma b^2c^2 \geqq 4s^2[ks^2+2k(k-5)Rr+(k+4)r^2]/(k+1)^2. \tag{6}$$

Remark 2.1. From (6), GI 5.8: $s^2 \geqq 16Rr - 5r^2$ and GI 5.1: $R \geqq 2r$ it follows that

$$\begin{aligned}\Sigma b^2c^2 &\geqq 4s^2[ks^2+2k(k-5)Rr+(k+4)r^2]/(k+1)^2\\ &\geqq 8s^2[k(k+3)Rr-2(k-1)r^2]/(k+1)^2 \geqq 16F^2.\end{aligned} \tag{6'}$$

Remark 2.2. The inequalities (6′) show that the $GT_I(k)$-dual inequality (6) is sharper than initial inequality GI 4.12.

Remark 2.3. By setting $k=1$ in (6) we get

$$\Sigma b^2 c^2 \geqq s^2(s^2 - 8Rr + 5r^2) \tag{7}$$

which immediately implies

$$\Sigma b^2 c^2 \geqq s^2(s^2 - 8Rr + 5r^2) \geqq 8s^2 Rr \geqq 16F^2. \tag{7'}$$

Example 3. GI 5.13: $\Sigma a^2 \leqq 9R^2 \xrightarrow{GT_I(k)}$

$$\begin{aligned}&(k+1)r[2(k-1)(7k-9)R^2 - (k^2+17k-16)Rr + 4r^2]\\ &\leqq (k-1)[k(k-1)R - 4r]s^2.\end{aligned} \tag{8}$$

Remark 3.1. From (8) we obtain GI 5.1 (by setting $k=1$), which is equivalent to GI 5.20: $\Sigma a(s-a) \leqq 9Rr$.

Remark 3.2. We will not consider the case $1<k<2$ because then the multiplier $k(k-1)R - 4r$ in the right-hand side of (8) has variable sign.

Remark 3.3. From (8) (for $k=2$), we get

$$s^2 \geqq 15Rr - 3r^2,$$

which is weaker than the left inequality of GI 5.8.

Remark 3.4. The inequality (8) gives a lower bound for s^2 which is a rational function of k, R, r $(k>2)$:

$$\begin{aligned}&s^2 \geqq r(k+1)[2(k-1)(7k-9)R^2 - (k^2+17k-16)Rr + 4r^2]/\\ &(k-1)[k(k-1)R - 4r] = f(k,R,r).\end{aligned} \tag{9}$$

It is interesting to find the value of k for which we get the best possible inequality of type (9).

Case 4.1. For $k=3$, (9) yields

$$s^2 \geqq 4r(12R^2 - 11Rr + r^2)/(3R - 2r) = f(3,R,r), \tag{10}$$

which is sharper than the best possible quadratic inequality GI 5.8 for s^2, i.e.

$$s^2 \geqq 4r(12R^2 - 11Rr + r^2)/(3R - 2r) \geqq r(16R - 5r). \tag{10'}$$

Case 4.2. For $k \geqq 3$ we will prove the inequality

$$f(3, R, r) \geqq f(k, R, r), \tag{11}$$

which is equivalent to following

$$(k-3)(x-2)\,\varphi(k, x) \geqq 0, \tag{12}$$

where $x = R/r \geqq 2$ and

$$\begin{aligned}\varphi(k, x) &= x(6x-1)k^2 - x(24x-19)k + 2(9x^2-12x+2)\\ &= 6(k-1)(k-3)x^2 + (-k^2+19k-4)x + 4.\end{aligned} \tag{13}$$

The inequality (12) is true, since

$$\begin{aligned}\frac{\partial\varphi(k, x)}{\partial k} &= 2x(6x-1)k - x(24x-19) \geqq 2x(6x-1)3 - x(24x-19)\\ &= 12x^2 + 15x > 0\end{aligned}$$

and hence

$$\begin{aligned}\varphi(k, x) \geqq \varphi(3, x) &= 9x(6x-1) - 3x(24x-19) + 2(9x^2-12x+2)\\ &= 24x + 4 \geqq 52 > 0.\end{aligned}$$

Consequently, from (10)–(12) we have

$$s^2 \geqq f(3, R, r) \geqq f(k, R, r) \quad \text{for} \quad k \geqq 3. \tag{14}$$

Case 4.3. For $k \in (2, 3)$ we will determine the sign of the function $\varphi(k, x)$ for $x \geqq 2$. From $\varphi(k, x) = 0$ we get

$$x_{1,2} = [p \pm (p^2 + 96q)^{1/2}]/(-12q) = 8/[p \mp (p^2 + 96q)^{1/2}], \tag{15}$$

where

$$p = k^2 - 19k + 24, \qquad q = (k-1)(3-k) > 0.$$

The roots x_1 and x_2 from (15) belong respectively to the intervals $(-\frac{1}{3}, -\frac{1}{6})$ and $(2, +\infty)$. Indeed, the inequalities $-\frac{1}{6} < x_1 < -\frac{1}{3}$ are equivalent to

$$p + 24 < (p^2 + 96q)^{1/2} < p + 48,$$

which together with $p + 24 = (3-k)(16-k) > 0$ imply the obvious inequalities

$$-(k-2)(7-k) < 0 < (3-k)(17-2k).$$

Also the inequality $2<x_2$ is equivalent to

$$(p^2+96q)^{1/2}<4-p\,,$$

and taking into account that

$$-k^2+19k-34=(17-k)(k-2)>0\,,$$

i.e.

$$0<14<-k^2+19k-20=4-p\,,$$

we obtain the obvious inequality

$$0<(k-2)(11k-7)\,.$$

Moreover it is not difficult to prove that:

$$\lim_{k\to 2^+} x_1=-\tfrac{1}{3}\,,\qquad \lim_{k\to 2^+} x_2=2\,,\qquad \lim_{k\to 3^-} x_1=-\tfrac{1}{6}\,,\qquad \lim_{k\to 3^-} x_2=+\infty\,.$$

From all the above, it follows that

$$\varphi(k,x)\geqq 0 \quad \text{for} \quad 2\leqq x\leqq x_2\,, \quad 2<k<3$$

and

$$\varphi(k,x)\leqq 0 \quad \text{for} \quad x_2\leqq x<+\infty\,, \quad 2<k<3\,.$$

Therefore, the inequality (11) or its equivalent (12) holds, for $x_2\leqq x<+\infty$, when $k\in(2,3)$, but not for all $x\geqq 2$. For example:

i) if $k=2{,}001$ then (11) holds for $x\geqq 2{,}0021452\ldots$
ii) if $k=2{,}999$ then (11) holds for $x\geqq 2000,\ 5001\ldots$.

Consequently, in the case $k\in(2,3)$, we may not say anything about the truth of the inequality (11) without further restrictions on $x=R/r\geqq 2$.
And now, if we suppose there exists $k\in(2,3)$ such that the inequality

$$f(k,R,r)\geqq f(3,R,r) \tag{16}$$

holds for all $x\geqq 2$, we get a contradiction, because for each $k\in(2,3)$ there always exists a $x_2\geqq 2$ such that the inequality (11) (which is the inverse inequality of (16)) is realized for all $x\geqq x_2$.

Remark 3.5. By setting in (8) $k=1+m$, $m\geqq 0$, we get

$$(m+2)\,r\,[2m(7m-2)R^2-(m^2+19m+2)Rr+4r^2]\leqq m[m(m+1)R-4r]\,s^2\,. \tag{17}$$

Then for $m \geqq 1$ we have

$$s^2 \geqq (m+2)\, r\, [2m(7m-2)\, R^2 - (m^2+19m+2)\, R\, r + 4r^2]/m\, [m(m+1)\, R - 4r]\,. \tag{17'}$$

The same inequality (17′) is obtained for $0 \leqq m \leqq 1$ and $R > 4r/m(m+1)$. In the case, when $0 \leqq m \leqq 1$ and $2r \leqq R < 4r/m(m+1)$, we obtain

$$s^2 \leqq (m+2) r [2m(7m-2)\, R^2 - (m^2+19m+2)\, R\, r + 4r^2]/m\, [m(m+1)\, R - 4r]\,. \tag{17''}$$

Finally, we note that the inequality (10) gives the best presently known rational lower bound for s^2, because of the following chain of well-known inequalities

$$\begin{aligned} s^2 &\geqq 4r(12R^2 - 11Rr + r^2)/(3R-2r) \geqq r(16R-5r) \geqq r(4R+r)^2/(R+r) \\ &\geqq r(16R+3r)(4R+r)^2/(4R-r)(4R+7r) \geqq r(4R+r)^3/(2R-r)(2R+5r) \\ &\geqq 3r(4R+r) \geqq r(4R+r)^2/(2R-r) \geqq 3r(4R+r)^3/(7R-5r)^2\,. \end{aligned} \tag{18}$$

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Funktionen beschränkter Homogenität

Wir führen in dieser Arbeit eine neue Funktioneneigenschaft ein, die zur Unterscheidung vom gewöhnlichen Homogenitätsbegriff (Gl. 1–3) als «beschränkte Homogenität» (Gl. 5–6) bezeichnet wird. Anschliessend veranschaulichen wir diese spezielle Skaleneigenschaft durch mehrere praktische Beispiele (Gl. 8–23).

Wir betrachten hier stetig differenzierbare Funktionen $f\colon R^m \to R$ wobei $R^m = \{X = (x_1, x_2, \ldots, x_m) \mid x_k \in R\}$. Einfachheitshalber schreiben wir aber fast alle Gleichungen nur für $m = 2$ oder $m = 1$ auf, weil die Verallgemeinerung für beliebiges m offensichtlich ist.

Nach der gewöhnlichen Definition ist $f = f(X)$ eine homogene Funktion vom Homogenitätsgrad s, wenn

$$f(\lambda x_1, \lambda x_2) = \lambda^s f(x_1, x_2) \tag{1}$$

für beliebiges $\lambda \in R^+$ erfüllt ist.