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Autor(en): **McMullen, P.**

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The order of a finite Coxeter group

Abstract. The Brianchon-Gram theorem leads to purely elementary calculations for the order of the finite orthogonal groups generated by reflexions in hyperplanes, and for the densities of the regular 4-dimensional star-polytopes.

1. Introduction

Let G be a finite group generated by reflexions in hyperplanes or *mirrors* in n-dimensional euclidean space E^n. These hyperplanes must contain a common point; if we take this point to be the origin o of coordinates, then G is an orthogonal group.

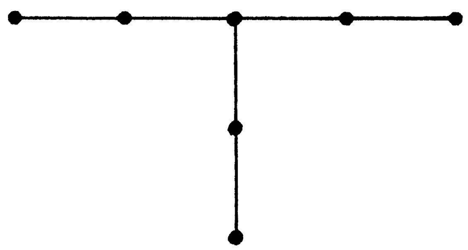
The images under G of the mirrors of the generated reflexions R_j dissect the space E^n into congruent convex cones, which are fundamental regions for G, and whose number is obviously the order $|G|$ of G. Thus, to find $|G|$, it is only necessary to count these cones, or, equivalently, measure their normalized volumes or *angles*. However, therein lies a problem, since if $n \geq 4$, no strictly elementary way of doing this has been available hitherto.

Let us assume that G is *irreducible*, that is, acts irreducible on E^n. Then it is well known (see [3, Chapter 11]) that G is generated by precisely n reflexions, whose mirrors may be chosen to bound any one of the fundamental cones in the dissection of E^n just described. If these mirrors are $H_1, \ldots, H_n$, then for $1 \leq j < k \leq n$, the dihedral angle between H_j and H_k is π/p_{jk} for some integer $p_{jk} \geq 2$, and if R_j is the reflexion in H_j ($j = 1, \ldots, n$), then G has the presentation

$$G = \langle R_1, \ldots, R_n \mid (R_j R_k)^{p_{jk}} = E (1 \leq j \leq k \leq n) \rangle ,$$

where $p_{jj} = 1$ for $1 \leq j \leq \leq n$, and E is the identity. Abstract groups with such presentations are known as *Coxeter groups*; Coxeter has shown (see [22]) that all finite Coxeter groups are, in fact, isomorphic to reflexion groups (there is actually a more general result than this, but we only need this case).

It is useful to denote G by its *Coxeter diagram* (see [3, 11.3]), which is a graph with a *node* corresponding to each reflexion R_j or mirror H_j, with nodes j and k joined by a *branch* labelled p_{jk} if the dihedral angle between H_j and H_k is π/p_{jk}; this formulation will later permit fractional labels $p_{jk} > 1$. It is customary to omit branches labelled 2, and, because of their frequency, to omit labels 3 on branches. The condition that G be irreducible is just that its Coxeter diagram be connected; if G is reducible, then it is the direct product of the reflexion groups corresponding to the connected components of the Coxeter diagram. As an example (which we shall use in Section 3 below), the Coxeter diagram of the infinite group $T_7 = [3^{2,2,2}]$ is



Let us briefly survey the currently known methods for calculating the order $|G|$ of G. First, we may associate with G a convex polytype, the numbers of whose faces of various kinds are the indices $[G:H]$ of certain subgroups H of G which are also generated by reflexions. If n is odd, then Euler's theorem (see, for example, [5, Chapter 8]) and the knowledge of the orders $|H|$ will yield $|G|$ (see also the remarks in Section 4 below), so that the actual polytope need not be constructed. When n is even, so that Euler's theorem on its own can only yield the ratios of the numbers of faces, a suitable polytope can often be constructed by synthetic methods, and again the value of $|G|$ results. (From an historical point of view, of course, reflexion groups arise from polytopes, rather than other way round.) In fact, various simplex dissection results (for examples, see Section 5 below; see also [4], which contains many other useful references), enable us to avoid such arguments in all but a very few cases; unfortunately, such cases are the most interesting.

Second, for $n=4$, the order of the symmetry group of a regular polytope (which excludes only one of the five cases, and this is in any event a subgroup of index 2 in one of the others) can be calculated with the aid of a solution of a certain trigonometrical equation (see Section 6). An alternative method involves the evaluation of certain integrals due to Schläfli; since the most relevant one of these cannot be evaluated directly, recourse must be had once more to the simplex dissection results (again, see [4]).

Finally, for other even $n \geq 6$ ($n=6$ and $n=8$ are the only important cases), the group G can be associated with a honeycomb, and $|G|$ can be found from the relative numbers of faces of this honeycomb. However, none of these last three methods is elementary; in particular, the last depends upon the somewhat deep result (see [3, 9.8]) that these relative numbers exist, and that the analogue of Euler's theorem holds for them. (A variant of this technique appears in [1], but it is used there with knowledge of the order of the group [3, 3, 5] to calculate the densities of the regular 4-dimensional star-polytopes. It is described in [3] as resting «on rather flimsy foundations»; however, see Section 5 below. The Schäfli function provides an alternative approach to calculating the densities; see [4] once more.)

The method which we shall describe here is, in a vague sense, related to the last of these methods (we shall make the connexion more explicit later), but the result to which we shall appeal (the Brianchon-Gram theorem) relies only on the ordinary Euler theorem (in E^{n-1}). We shall give this result, and a closely related one, in Section 2, and then apply them to our problem of determining the orders of reflexion groups in the following Sections.

It is worth making an additional remark at this stage. Our calculations will be purely geometric; in other words, though we often use the language of group theory, we do not really make use of the fact that the cones whose sizes we find are the fundamental regions of groups. What we do use is the fact that certain hyperplanes, with given angles between them, determine either simplicial cones or finite euclidean simplices; the criteria which must be satisfied (the Schäfli determinant condition, for which see [3, 7.7]; we shall take its particular applications for granted) pay no regard to whether the reflexions in these hyperplanes generate a finite or discrete group. We shall refer to this again several times later.

2. Two angle-sum relations

Let K be an n-dimensional polyhedral set in E^n, and let F be a non-empty face of K. The *(inner) angle* $\alpha(F, K)$ of K at F is that proportion of a sufficiently small ball centred at a relatively interior point x of F which lies in K:

$$\alpha(F, K) = \lim_{\varrho \to 0} \frac{V(B(x, \varrho) \cap K)}{V(B(x, \varrho))},$$

where $B(x, \varrho)$ is the ball of radius $\varrho > 0$ with centre x, and V denotes volume. We shall employ two angle-sum relations:

Theorem 1 (Brianchon-Gram). *If $n \geq 1$ and P is an n-polytope in E^n, then*

$$\sum_F (-1)^{\dim F} \alpha(F, P) = 0.$$

Theorem 2 (Sommerville). *If C is a polyhedral cone in E^n, then*

$$\sum_F (-1)^{\dim F} \alpha(F, C) = (-1)^n \alpha(A, C),$$

where A is the face of apices of C.

In both theorems, the sums extend over all non-empty faces F.
A common generalization of these theorems to arbitrary polyhedral sets is proved in [7]; it is perhaps worth noting that there it is made clear that the results hold on the level of equidissectability (in Sommerville's theorem, the cone C must be replaced by its negative on the right of the equation). Further details about the background to these results can be found there and in [5, 14.1]; for an easy proof of Theorem 1, see [8]. In our applications, a cone C will always be pointed (with a single apex).

3. The crystallographic groups

We call a reflexion group G *crystallographic* if G is a subgroup of an infinite discrete group $\tilde{G}$ generated by reflexions (in the same space). Similar considerations to those in Section 1

apply, and the mirrors of all the reflexions in $\tilde{G}$ dissect E^n into fundamental regions, which are simplices if $\tilde{G}$ (or even G) is irreducible. We shall apply Theorem 1 to these simplices. Let T be such a simplex, bounded by the hyperplanes $H_0, \ldots, H_n$, which are the mirrors of the reflexions $R_0, \ldots, R_n$. We shall always be able to suppose that $G = \langle R_1, \ldots, R_n \rangle$, and that any subgroups generated by other proper subsets of the R_j are isomorphic to subgroups of G (that is, that G is the *special subgroup* of $\tilde{G}$ in the sense of [3, p. 191]); some of these subgroups may actually be isomorphic to G itself. In any event, for each proper subset $J \subset N := \{0, \ldots, n\}$, $G_J := \langle R_j \mid j \in J \rangle$ is a finite subgroup of $\tilde{G}$, which leaves invariant the flat $H_J := \bigcap \{H_j \mid j \in J\}$, and hence the face $T_J := H_J \cap T$ of T. (The discreteness of $\tilde{G}$ ensures that all these subgroups G_J are actually finite, but we emphasize once again that we shall be performing pure angle calculations, which do not depend on this discreteness.)
Let us write $g_J := |G_J|$. The discussion of Section 1, and the definition of angle in Section 2, show at once:

Lemma 1. *For each $J \subset N$,*

$$\alpha(T_J, T) = \frac{1}{g_J}.$$

Note here that for the case $J = \emptyset$, we have $T_\emptyset = T$ and $g_\emptyset = 1$. Theorem 1 and Lemma 1 yield at once:

Theorem 3. *Let $\tilde{G}$ be an irreducible discrete infinite reflexion group in E^n, with generating reflexions $R_0, \ldots, R_n$ in the bounding hyperplanes of its fundamental region T. For each $J \subset N$, let $G_J := \langle R_j \mid j \in J \rangle$, and let $g_J := |G_J|$. Then*

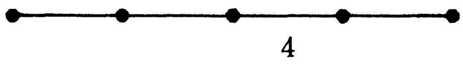
$$\sum_J \frac{(-1)^{\operatorname{card} J}}{g_J} = 0.$$

We shall only give three examples of this result, because although its character is elementary, the calculations involved in its application are lengthy (in n-dimensions, there are $2^{n+1} - 1$ terms in the expression above). The interested reader will easily determine which infinite discrete reflexion group has a given finite crystallographic reflexion group as its special subgroup (the list in [3, Table IV] is arranged to make this straightforward). Let us find the orders of the groups [3, 3, 4], [3, 4, 3] and E_6 (we use the notation of [3] here and elsewhere). (We should remark that the order of [3, 3, 4] can be found more simply with the aid of the generic simplex dissection results we shall discuss later; [3, 4, 3] can be dealt with by another generic simplex dissection which we shall not need here, but E_6 provides a problem of a deeper kind.)
First, [3, 3, 4] is the special subgroup of the group [4, 3, 3, 4]. Applying Theorem 3 to the simplex

yields for the order g:

$$\frac{2}{g}+\frac{2}{96}+\frac{1}{64}-\frac{2}{48}-\frac{1}{24}-\frac{4}{16}-\frac{2}{12}-\frac{1}{8}+\frac{2}{8}+\frac{2}{6}+\frac{6}{4}-\frac{5}{2}+1=0,$$

(We have substituted in the (assumed known) orders of the lower-dimensional groups, and used the fact that disconnected Coxeter graphs correspond to direct products; see Section 1.) From this easily follows $g=384$.
In turn, [3, 4, 3] is the special subgroup of [3, 3, 4, 3]. Theorem 3 applied to the simplex



yields for the order g:

$$\frac{1}{g}+\frac{1}{384}+\frac{1}{96}+\frac{1}{36}+\frac{1}{48}-\frac{2}{48}-\frac{1}{24}-\frac{1}{16}-\frac{5}{12}-\frac{1}{8}+\frac{1}{8}+\frac{3}{6}+\frac{6}{4}-\frac{5}{2}+1=0,$$

from which follows $g=1152$.
Finally, we come to $E_6=[3^{2,2,1}]$. This is the special subgroup of $T_7=[3^{2,2,2}]$ (see Section 1 for its diagram), and assuming that we have already found the orders of the lower-dimensional groups ($|A_k|=(k+1)!$ and $|B_k|=2^{k-1}\cdot k!$ for $k=4$ or 5 are the only extra orders we need; see [3, Table IV]), the order g satisfies

$$\frac{3}{g}+\frac{3}{1440}+\frac{1}{216}-\frac{3}{720}-\frac{6}{72}-\frac{3}{1920}-\frac{6}{240}-\frac{3}{96}+\frac{6}{120}+\frac{3}{36}+\frac{1}{192}+\frac{15}{24}+\frac{9}{48}+\frac{1}{16}$$
$$-\frac{6}{24}-\frac{18}{12}-\frac{11}{8}+\frac{6}{6}+\frac{15}{4}-\frac{7}{2}+1=0,$$

which yields $g=72.6!$.

4. The odd dimensional case

If the dimension n is odd, then Theorem 2 gives us an alternative approach, because for an n-dimensional pointed polyhedral cone C with apex o, it states

$$\alpha(\{o\},C)=\frac{1}{2}\sum_{F\neq\{o\}}(-1)^{n-\dim F}\,\alpha(F,C).$$

(When n is even, the terms $\alpha(\{o\},C)$ in which we are interested cancel.) The implication for groups is:

Theorem 4. *Let n be odd, and let $G=\langle R_1,\dots,R_n\rangle$ be a finite reflection group of order $g=|G|$. For each $J\subset N=\{1,\dots,n\}$ let $G_J=\langle R_j\,|\,j\in J\rangle$ and $g_J=|G_J|$. Then*

$$\frac{1}{g}=\frac{1}{2}\sum_{J\subset N}\frac{(-1)^{\operatorname{card}J}}{g_J}.$$

For example, if $n=3$, p, q are integers, and $G_{p,q}$ has Coxeter diagram

then $g_{p,q}=|G_{p,q}|$ is given by

$$\frac{1}{g_{p,q}}=\frac{1}{2}\left\{\frac{1}{2p}+\frac{1}{2q}+\frac{1}{4}-\frac{3}{2}+1\right\}$$
$$=\frac{1}{4}\left\{\frac{1}{p}+\frac{1}{q}-\frac{1}{2}\right\},$$

a formula familiar from [3, 5.43]. We stress here that, if we interpret the left side of this expression as an angle, we do not actually have to assume that p and q are integers (or even rationals).

5. The group [3, 3, 5]

The non-crystallographic reflexion groups were not dealt with in the preceding sections. However, the only group which actually escaped our treatment was [3, 3, 5], the group of the regular 120-cell (or 600-cell) in E^4. We shall now see that even this group is amenable to our approach; as a bonus, we shall also be able to calculate the densities of the regular star-polytopes in E^4. (The precise geometric meaning of *density* is defined in [3, page 94], and we shall not concern ourselves with it overmuch. In any event, we are performing pure angle computations.)
For convenience, let us denote by $S(p_1, \ldots, p_{n-1})$ the simplicial cone or simplex

and let $\alpha(p_1, \ldots, p_{n-1})$ denote its angle (which is taken as 0 for a euclidean simplex). (We shall only use the cases $n=4$ or 5.) Our basic result, which is just Theorem 1 with substitution for the angles of products of cones of dimension at most 3, is:

Theorem 5. *Let $S(p, q, r, s)$ be a euclidean 4-simplex. Then*

$$\alpha(p, q, r)+\alpha(q, r, s)=\frac{1}{8}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s}-\frac{2}{ps}-1\right).$$

(The result of [1] which Coxeter held in [3, Chapter XIV] to «rest on rather flimsy foundations» is basically this, interpreted in terms of orders of groups and densities.)
We now apply Theorem 5 to various simplices arising in the dissection of E^4 by the fundamental cones of [3, 3, 5]. But first we need some simplex dissection results; we refer to these as *generic*, because they do not depend on our working in any particular group (in fact, the simplices involved do not have to correspond to any group). Those we use all occur in [4].

Theorem 6. *Let $p>2$. Then*

a) $\alpha\left(3, p, \frac{p}{2}\right) = 4\alpha(3, 3, p)$.

b) $\alpha\left(p, \frac{p}{2}, p\right) = 6\alpha(3, 3, p)$.

To see this, we merely observe that the simplicial cone whose dihedral angles are all $2\pi/p$ can be dissected into 24 cones $S(3, 3, p)$, 6 cones $S\left(3, p, \frac{p}{2}\right)$, or 4 cones $S\left(p, \frac{p}{2}, p\right)$. The result follows at once. □

As a useful convention, whenever $p>1$ we shall define p' by

$$\frac{1}{p}+\frac{1}{p'}=1.$$

Then we have

Theorem 7. *Let C be the simplicial cone in E^4 with dihedral angles π/p_{ij} $(1 \leq j<k \leq 4)$, and let C' be the cone obtained by replacing p_{j4} by p'_{j4} for $j=1, 2, 3$. Then*

$$\alpha(C)+\alpha(C')=\frac{1}{4}\left\{\frac{1}{p_{12}}+\frac{1}{p_{13}}+\frac{1}{p_{23}}-1\right\}.$$

The two cones C and C' fit together along their common 4th face, to form the product of a line with a 3-dimensional cone D whose dihedral angles are π/p_{ij} for $1 \leq j<k \leq 3$. Thus $\alpha(C)+\alpha(C')$ is the angle of this product cone, which is just that of D, and so the number given in the theorem. □

We shall need two consequences of Theorem 7.

Corollary 7.1

$$\alpha(p, q, r')=\frac{1}{4}\left\{\frac{1}{p}+\frac{1}{q}-\frac{1}{2}\right\}-\alpha(p, q, r).$$

Corollary 7.2

$$\alpha(p, q', r)=\alpha(p, q, r)+\frac{1}{4}\left\{\frac{1}{2}-\frac{1}{q}\right\}.$$

Corollary 7.1 is a direct application of Theorem 7, which also yields

$$\alpha(p, q', r)=\frac{1}{4}\left\{\frac{1}{p}+\frac{1}{2}+\frac{1}{2}-1\right\}-\alpha(p, q, r'),$$

and Corollary 7.2 follows at once from Corollary 7.1. □

We now employ these results to calculate $g := |[3, 3, 5]|$ and the densities of the related regular star-polytopes. We begin by defining two numbers d_1 and d_2 by

$$\alpha(\tfrac{5}{2}, 3, 5) = d_1 \alpha(3, 3, 5),$$
$$\alpha(3, 3, \tfrac{5}{2}) = d_2 \alpha(3, 3, 5).$$

It is worth stressing at this point that we shall make no prior assumption that d_1 or d_2 is even rational, let alone an integer. (Of course, [6] shows that they must both be integers, because the regular star-polytopes $\{\tfrac{5}{2}, 3, 5\}$ and $\{3, 3, \tfrac{5}{2}\}$ have the same symmetry group [3, 3, 5] as $\{3, 3, 5\}$, but that paper is completely independent of the present one.) We shall take for granted [3, 14.14], which we can read as saying that certain simplices are euclidean.

Since the simplex $S(\tfrac{5}{2}, 3, 3, 5)$ is euclidean, we have

$$\alpha(3, 3, \tfrac{5}{2}) + \alpha(3, 3, 5) = \frac{1}{8}\left\{\frac{2}{5} + \frac{1}{3} + \frac{1}{3} + \frac{1}{5} - \frac{4}{25} - 1\right\},$$

and hence

$$\frac{d_2 + 1}{g} = \frac{1}{75} = \frac{192}{14\,400}.$$

Next, the simplex $S(3, \tfrac{5}{2}, 5, 3)$ is euclidean, and so

$$\alpha(3, \tfrac{5}{2}, 5) + \alpha(3, 5, \tfrac{5}{2}) = \frac{1}{8}\left\{\frac{1}{3} + \frac{2}{5} + \frac{1}{5} + \frac{1}{3} - \frac{2}{9} - 1\right\} = \frac{1}{180}.$$

Now,

$$\alpha(3, 5, \tfrac{5}{2}) = 4\alpha(3, 3, 5) = \frac{4}{g}$$

from Theorem 6, while

$$\begin{aligned}\alpha(3, \tfrac{5}{2}, 5) &= \frac{1}{4}\left\{\frac{1}{3} + \frac{2}{5} - \frac{1}{2}\right\} - \alpha(3, \tfrac{5}{2}, \tfrac{5}{4}) \\ &= \frac{7}{120} - 4\alpha(3, 3, \tfrac{5}{2}) \\ &= \frac{7}{120} - \frac{4d_2}{g},\end{aligned}$$

from Corollary 7.1 and Theorem 6. Substituting, we have

$$\frac{d_2 - 1}{g} = \frac{1}{4}\left\{\frac{7}{120} - \frac{1}{180}\right\} = \frac{19}{1440} = \frac{190}{14\,400}.$$

We deduce immediately that $g=14\,400$ and $d_2=191$.
If we write

$$d(p, q, r) = g\,\alpha(p, q, r) = 14\,400\,\alpha(p, q, r)$$

for the density of $S(p, q, r)$ (for appropriate p, q, r), we have the obvious values

$$d(3, 5, \tfrac{5}{2}) = 4,$$

$$d(5, \tfrac{5}{2}, 5) = 6$$

from Theorem 6, while that result and Corollaries 7.1 and 7.2 yield

$$d(\tfrac{5}{2}, 5, \tfrac{5}{2}) = 14\,400\left\{\alpha(\tfrac{5}{2}, \tfrac{5}{4}, \tfrac{5}{2}) + \frac{1}{4}\left[\frac{1}{2} - \frac{4}{5}\right]\right\} = 6d_2 - 1080 = 66,$$

$$d(3, \tfrac{5}{2}, 5) = 14\,400\,\frac{7}{120} - 4d_2 = 76,$$

where in the last equation we have not repeated our previous calculations.
It remains to find d_1. Since the simplex $S(\tfrac{5}{2}, 3, 5, \tfrac{5}{2})$ is euclidean, we have

$$\alpha(\tfrac{5}{2}, 3, 5) + \alpha(3, 5, \tfrac{5}{2}) = \frac{1}{8}\left\{\frac{2}{5} + \frac{1}{3} + \frac{1}{5} + \frac{2}{5} - \frac{8}{25} - 1\right\} = \frac{1}{600},$$

and hence

$$d_1 = 14\,400\,\frac{1}{600} - 4 = 20.$$

This completes our calculations. We may observe that we have not used the euclidean simplices $S(\tfrac{5}{2}, 5, \tfrac{5}{2}, 5)$ or $S(5, 3, \tfrac{5}{2}, 5)$, which yield no new information.

6. The Petrie polygon

We end by briefly discussing the trigonometric calculation for g when $n=4$ mentioned in Section 1. The length h of the Petrie polygon (see [3, 12.4] for the definition) of the regular convex polytope $\{p, q, r\}$ is the integer solution of the equation

$$\left(\cos^2\frac{\pi}{h} - \cos^2\frac{\pi}{p}\right)\left(\cos^2\frac{\pi}{h} - \cos^2\frac{\pi}{r}\right) = \cos^2\frac{\pi}{h}\cos^2\frac{\pi}{q}$$

(see [3, 12.35]; of course, such an equation holds when p, q, r are not integers, except that there may only be rational solutions h). However, h is also related to g by

$$\frac{64h}{g} = 12 - p - 2q - r + \frac{4}{p} + \frac{4}{r}$$

(see [3, 12.81]). In view of our independent calculations for g, this last equation yields an alternative way of finding h.

P. McMullen, University College London

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Kanonische Codierungen von $\mathbb{N}^k$

Als erster brauchte Georg Cantor eine Codierung von $\mathbb{N}^2$, d.h. eine Bijektion zwischen $\mathbb{N}$ und $\mathbb{N}^2$. (Sie entspricht dem weiter unten angegebenen $\langle\ \rangle_2$.) Es gelang ihm, damit zu beweisen, dass unendliche Mengen nicht durch endliches Anwenden des kartesischen Produktes auf sich selbst vergrössert werden können, sondern eben immer dieselbe Mächtigkeit behalten (siehe [1] oder [2]). Heute ist sein Verfahren, das *Cantor*sche Diagonalisierungsverfahren, allgemein bekannt und findet sich in vielen Lehrbüchern, die sich mit dem Aufbau der Zahlensysteme befassen (z.B. [3], [4]). Andere Codierungen von abzählbaren Mengen treten vielerorts in der Logik und Berechenbarkeitstheorie auf, zum Beispiel als Gödelnummern, wenn es darum geht, sowohl die Zahlen selbst, als auch die Funktionen, die man auf sie anwenden kann, in einem einzigen Bereich darzustellen. Ein neueres Beispiel dafür liefern Modelle für den λ-Kalkül [5], in denen die ganze Berechenbarkeitstheorie behandelt werden kann. Dieses letzte Beispel ist insofern bemerkenswert, als es eines der wenigen ist, in denen die explizite Form einer Bijektion zwischen $\mathbb{N}^2$ und $\mathbb{N}$ überhaupt eine Rolle spielt. Für das Standard-Modell wählt man gerade $\langle\ \rangle_2$.
Unter einer *Codierung* von $\mathbb{N}^k$ verstehen wir nun jede bijektive Abbildung $\mathbb{N}^k \to \mathbb{N}$, mithin eine Abbildung, die jedem k-Tupel $(n_1, \ldots, n_k)$ von $\mathbb{N}^k$ in eindeutiger Art und Weise eine Zahl c zuordnet, die wir als *Code* von $(n_1, \ldots, n_k)$ bezeichnen.