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A Geometric Property of Functions Harmonic in a Disk

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Abstract: An alternative way to compute the value of the Poisson integral of a function integrable on a circle is given, using line segments and averages. A theorem of classic geometry, about circles and chords, follows.

0 Introduction

The Dirichlet problem for the open unit disk $\mathbb{D}$ in $\mathbb{C}$ is the following: given a continuous function f on $\partial\mathbb{D}$, one wants to construct a function F on $\bar{\mathbb{D}}$ which is harmonic in $\mathbb{D}$ (i.e. twice continuously differentiable, with $\frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} = 0$), and such that $F = f$ on $\partial\mathbb{D}$.

Das Dirichlet-Problem (G.L. Lejeune Dirichlet 1805-1859) besteht darin, in einem Gebiet B eine harmonische Funktion f, also eine Funktion f mit $\Delta f = 0$ zu suchen, welche auf dem Rand ∂B des Gebietes B vorgegebene Werte annimmt. Diese Fragestellung, auch bekannt unter dem Namen "Erstes Problem der Potentialtheorie", bildete in der ersten Hälfte des letzten Jahrhunderts eine der ganz zentralen Fragestellungen der Analysis. Da bekanntlich ein enger Zusammenhang zwischen harmonischen Funktionen in der Ebene und analytischen Funktionen einer komplexen Variablen besteht, haben die Versuche zur Lösung des Dirichlet-Problems und das Studium der damit zusammenhängenden Fragen die Entwicklung der Analysis und der Funktionentheorie im letzten Jahrhundert von Riemann bis zu Hilbert stark beeinflusst. Für eine Kreisscheibe in der Ebene wird das Dirichlet-Problem durch das Poisson-Integral (S.D. Poisson 1781-1840) gelöst, von dem im vorliegenden Artikel die Rede ist. Das genauere Studium des Poisson-Integrals führt den Autor schliesslich zu Anwendungen in der euklidischen und hyperbolischen Geometrie. ust

The solution to this problem is obtained using as F the Poisson integral of f, i.e.

$$F(z) = \int_0^{2\pi} f(e^{it}) P_z^{(2)}(t) \frac{dt}{2\pi}$$

where $P_z^{(2)}(t)$ is the Poisson kernel for $\mathbb{D}$; the pointwise convergence of $F(re^{it})$ to $f(e^{it})$, as $r \to 1$, has been proven by H.A. Schwarz. The function F makes sense even when f is merely integrable; again F is harmonic in $\mathbb{D}$, and the pointwise radial convergence to f holds almost everywhere.

This paper deals with some properties of functions harmonic in a disk. The main result is formula (7), which relates the Poisson integral of a function f to the averages of f along appropriate segments.

In section 2 we prove a relation between Euclidean and hyperbolic geometry in $\mathbb{D}$, and in section 3 we relate one- and two-dimensional Poisson integrals.

1 Harmonic functions and line segments

Let $\mathbb{D}$ denote the open unit disk in $\mathbb{R}^2 \cong \mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\partial\mathbb{D}$ its boundary; we identify, for conveniency, the points $e^{it} \in \partial\mathbb{D}$ and $t \in \mathbb{T} = [-\pi, \pi)$. For any $z \in \mathbb{D}$, the Poisson kernel "centered" at z is

$$P_z^{(2)}(t) = \frac{1 - |z|^2}{|e^{it} - z|^2}.$$

For $z \in \mathbb{D}$, let $\ell(z, \cdot)\mathbb{T} \to \mathbb{T}$ be defined by the rule:

"the three points e^{it}, z and $e^{i\ell(z,t)}$ lie on the same line".

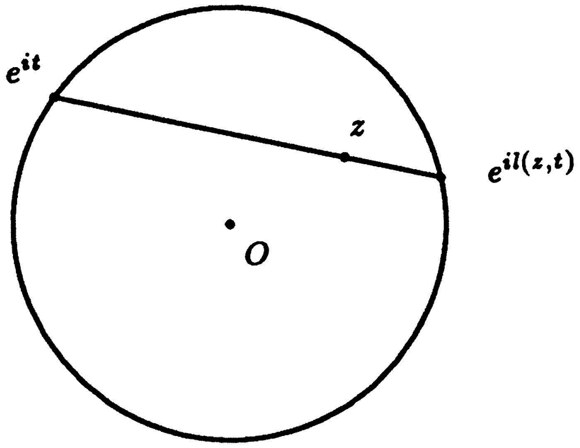

Fig. 1

This means that there exists some $0 < s(z,t) < 1$ such that:

$$z = [1 - s(z,t)]e^{it} + s(z,t)e^{i\ell(z,t)} \tag{1}$$

With the use of some analytic geometry it is easy to prove that

$$s(z,t) = \frac{1}{1 + P_z^{(2)}(t)}. \tag{2}$$

Proposition 1.1. *For $z \in \mathbb{D}$ and $t \in \mathbb{T}$ we have:*

$$\frac{d\ell(z,t)}{dt} = P_z^{(2)}(t); \tag{3}$$

$$P_z^{(2)}(t)\,P_z^{(2)}(\ell(z,t)) = 1. \tag{4}$$

Proof: The proof of (3) can be obtained differentiating (1) with respect to t; (4) follows from (3) and the identity $\ell(z,\ell(z,t)) = t$. □

If f is an integrable function on $(\mathbb{T}, dt)$, its (two-dimensional) Poisson integral in $\mathbb{D}$ is

$$(\mathcal{P}_D^{(2)} f)(z) = \int_{\mathbb{T}} f(e^{it})\,P_z^{(2)}(t)\,\frac{dt}{2\pi}; \tag{5}$$

$\mathcal{P}_D^{(2)} f$ is harmonic in $\mathbb{D}$, and it is the solution of the Dirichlet problem for $\mathbb{D}$, with boundary value f (see e.g. [**Ahl**], [**CBV**], or Vol.1 of [**Z**]).

Let

$$(\ell_z f)(t) = [1 - s(z,t)]f(e^{it}) + s(z,t) f(e^{i\ell(z,t)}) \tag{6}$$

be the weigthed average of the values of f at the endpoints of the chord joining e^{it}, z and $e^{i\ell(z,t)}$, i.e. the value obtained by linear interpolation between $f(e^{it})$ and $f(e^{i\ell(z,t)})$. Then:

Theorem 1.2. *For f integrable on $(\mathbb{T}, dt)$, and $z \in \mathbb{D}$:*

$$\int_{\mathbb{T}} (\ell_z f)(t)\,\frac{dt}{2\pi} = (\mathcal{P}_D^{(2)} f)(z) \tag{7}$$

$$\int_{\mathbb{T}} (\ell_z f)(t)\,P_z^{(2)}(t)\,\frac{dt}{2\pi} = (\mathcal{P}_D^{(2)} f)(z) \tag{7'}$$

Proof. If $\ell = \ell(z,t)$, from (3) and (4) we obtain $dt/d\ell = P_z^{(2)}(\ell)$; moreover, $s(z,t) + s(z,\ell) = 1$ for every t; thus

$$\begin{aligned}
\int_{\mathbb{T}} (\ell_z f)(t)\frac{dt}{2\pi} &= \int_{\mathbb{T}} \{[1 - s(z,t)]f(e^{it}) + s(z,t)f(e^{i\ell(z,t)})\}\,\frac{dt}{2\pi} \\
&= (\mathcal{P}_D^{(2)} f)(0) - \int_{\mathbb{T}} s(z,t)\,[f(e^{it}) - f(e^{i\ell})]\frac{dt}{2\pi} \\
&= (\mathcal{P}_D^{(2)} f)(0) + \int_{\mathbb{T}} [1 - s(z,\ell)]f(e^{i\ell})\,\frac{d\ell}{2\pi} - \int_{\mathbb{T}} s(z,t)f(e^{it})\,\frac{dt}{2\pi} \\
&= (\mathcal{P}_D^{(2)} f)(0) + \int_{\mathbb{T}} f(e^{i\ell})\,P_z^{(2)}(\ell)\,\frac{d\ell}{2\pi} - \int_{\mathbb{T}} s(z,t)\,[1 + P_z^{(2)}(t)]f(e^{it})\,\frac{dt}{2\pi} \\
&= (\mathcal{P}_D^{(2)} f)(0) + (\mathcal{P}_D^{(2)} f)(z) - (\mathcal{P}_D^{(2)} f)(0) \\
&= (\mathcal{P}_D^{(2)} f)(z).
\end{aligned}$$

The second equality has a similar proof. □

Formula (7) can be read as:

"The value at z of the harmonic function generated by f can be obtained computing the (weighted) averages of the values of f at the endpoints of all the chords passing through z, and then taking the mean value of these averages."

Remark 1.3. If f is continuous on $\mathbb{T}$, and $z = re^{iu}$, it is easy to see that the function $(\ell_z f)(\cdot)$ converges uniformly to the constant function $f(e^{iu})$, as $r \to 1^-$ and u is fixed. Thus we recapture the classical result (Schwarz' theorem) on the radial convergence of $\mathcal{P}_D^{(2)} f$ to f, just by taking the limit under the integral sign in (7) (see also [**M**]).

Remark 1.4. for $0 \le r < 1$, applying (3) and integrating from 0 to $t \in [0, \pi]$:

$$\int_0^t \mathbf{P}_r^{(2)}(t)\,dt = \ell(r,t) - \ell(r,0) = \ell(r,t) + \pi.$$

From (1) it follows that

$$\cos(\ell(r,t)) = \frac{2r - (1+r^2)\cos t}{1 + r^2 - 2r\cos t} = \frac{r' - \cos t}{1 - r'\cos t}, \quad \text{where } r' = \frac{2r}{1+r^2};$$

thus:

$$\int_0^t \mathbf{P}_r^{(2)}(t)\,dt = \pi - \arccos\left(\frac{r' - \cos t}{1 - r'\cos t}\right), \quad 0 \le t \le \pi.$$

This is an alternative formula to the more commonly used:

$$\int_0^t \mathbf{P}_r^{(2)}(t)\,dt = 2(1 - r'^2)^{-1/2} \arctan\left[\left(\frac{1+r'}{1-r'}\right)^{1/2} \tan\frac{t}{2}\right].$$

2 Relations with hyperbolic geometry

For $z \in \mathbb{D}$, we define the function $\gamma(z, \cdot) : \mathbb{T} \to \mathbb{T}$ by the following rule:

"the unique circle through z, orthogonal to $\partial\mathbb{D}$ in e^{it}, intersects $\partial\mathbb{D}$ in $e^{i\gamma(z,t)}$ ($\neq e^{it}$)."

In terms of hyperbolic geometry, e^{it} and $e^{i\gamma(z,t)}$ are the endpoints of the unique geodesic path joining e^{it} and z.

Since every Möbius map $w = g(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, $|b| < |a|$, preserves geodesics, it is immediate to show the "circle" version of (7′):

$$\int_{\mathbb{T}} f(e^{it})\,\mathbf{P}_z^{(2)}(t)\,\frac{dt}{2\pi} = \int_{\mathbb{T}} (\gamma_z f)(t)\,\mathbf{P}_z^{(2)}(t)\,\frac{dt}{2\pi},$$

where

$$(\gamma_z f)(t) = \frac{1}{2}[f(e^{it}) + f(e^{i\gamma(z,t)})].$$

Euclidean and non-euclidean geodesics are related by:

Proposition 2.1. *For* $z \in \mathbb{D}$, $t \in \mathbb{T}$ *and* $z' = \frac{2z}{1+|z|^2}$:

$$\gamma(z,t) = \ell(z',t). \tag{8}$$

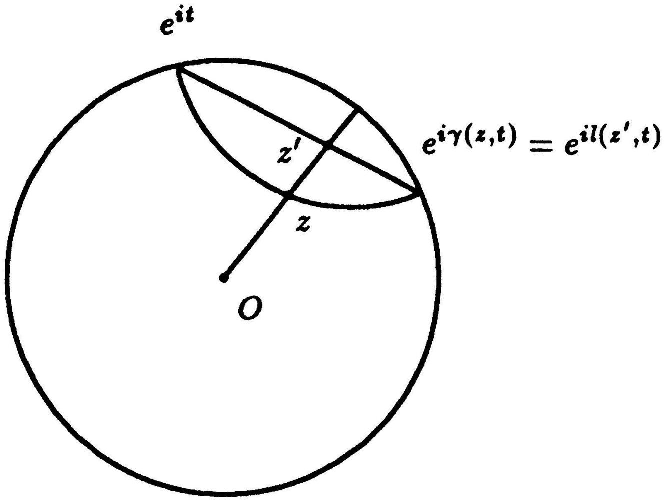

Fig. 2

The proof can be obtained after a tedious work of computation. However, when stated in terms of classic geometry, it seems to us an interesting result:

Given a circle Γ *and a point* A *interior to* Γ, *we associate to each point* P *on* Γ *the point* $P' \in \Gamma$ *such that the circle through* P, A *and* P' *intersects* Γ *orthogonally. All the chords joining pairs of associated points* P *and* P' *have a point in common.*

This fact can be proven noticing that the map associating P' to P is an involution; thus, straight lines joining corresponding points pass through the center of the involution.

It would be interesting to produce a proof involving only elementary geometry.

3 Iteration of Poisson integrals

We now look in a different way at formula (7).

The one-dimensional analogue of the unit disk $\mathbb{D}$ in $\mathbb{R}^2$ is the interval $I = (-1; 1)$, and the corresponding (one-dimensional) Poisson kernel is given by $P_x^{(1)}(\pm 1) = 1 \pm x$; similarly, the one-dimensional Poisson integral, relative to I, of a function f defined on $\partial I = \{\pm 1\}$, is given by:

$$(\mathcal{P}_I^{(1)} f)(x) = \frac{1-x}{2} f(-1) + \frac{1+x}{2} f(1).$$

Thus, if $L(z,t)$ denotes the chord of $\mathbb{D}$ with endpoints e^{it} and $e^{i\ell(z,t)}$, the value $(\ell_z f)(t) = [1-s(z,t)]f(e^{it}) + s(z,t)f(e^{i\ell(z,t)})$ is the one-dimensional Poisson integral $(\mathcal{P}^{(1)}_{L(z,t)} f)(x)$. Formula (7) becomes:

$$(\mathcal{P}_D^{(2)} f)(z) = ((\mathcal{P}_D^{(2)} \circ \mathcal{P}^{(1)}_{L(z,\cdot)}) f)(0), \tag{12}$$

showing that the two-dimensional Poisson integral can be written as average of the one-dimensional Poisson integrals.

Remark 3.1: this result can be extended to higher dimensions. It can be proven ([**Vi**]) that the $(N+1)$-dimensional Poisson integral of a function f, relative to the unit sphere B of $\mathbb{R}^{N+1}$, is the average of the N-dimensional Poisson integrals of f, relative to the N-spheres obtained by cutting B with all the possible hyperplanes passing through a fixex point of B. The proof of this fact is very technical.

References:

[**Ahl**] L.V. Ahlfors, Complex Analysis, McGraw Hill, 1979

[**CBV**] R.V. Churchill, J.W. Brown, R.F. Verhey, Complex variables and applications, McGraw Hill, 1974.

[**Z**] A. Zygmund, Trigonometric series, Cambridge Un. Press, 1959.

[**M**] D. Minda, The Dirichlet problem for a disk. Amer. Math. Monthly 97 (3) 1990, 220–223.

[**Vi**] M. Vignati, On the factorization of the Poisson integrals on spheres, preprint.

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