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The Magic World of Geometry — II. Geometry and Algebra of Braids.

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Vagn Lundsgaard Hansen received his M.Sc. in mathematics and physics from the University of Aarhus, Denmark in 1966 and his Ph.D. in mathematics from the University of Warwick, England in 1972. Since 1980 he is professor of mathematics at The Technical University of Denmark. He has published research papers in topology, geometry and global analysis, and the books *Braids and Coverings* (1989) and *Geometry in Nature* (1993). Also, he was the editor of the *Collected Mathematical Papers of Jakob Nielsen* (1986). He enjoys philosophical discussions, music and family life.

There is interesting mathematics even in the most common objects of daily life. In the second article of this series I shall show how geometry and algebra play together in the mathematical theory of braids.

Braids are among the oldest inventions of mankind. They are used for practical purposes to make rope, and for decorations in weaving patterns and hairstyles, etc. As mathe-

Knoten und — etwas allgemeiner — sogenannte Zöpfe gehören offensichtlich zu den einfachsten dreidimensionalen geometrischen Gebilden, die es gibt. Gerade aus diesem Grunde fordern sie die Mathematik zu einer abstrakten Behandlung heraus: Die Mathematik sollte eigentlich ein Verfahren liefern können, welches die Frage beantwortet, ob sich ein gegebener Knoten auflösen lässt oder nicht. Erste nennenswerte Fortschritte in diesem Problem wurden um 1930 von J.W. Alexander und von E. Artin erzielt. Dabei spielte die abstrakte Gruppentheorie eine zentrale Rolle. Auf diese Artinsche Zopftheorie geht V.L. Hansen im vorliegenden Beitrag, dem zweiten in der Reihe *The magic world of geometry*, näher ein. — Nach vielen Jahren mit nur kleinen Fortschritten hat das Gebiet der Knotentheorie neuerdings wieder stark an Interesse und Aufmerksamkeit gewonnen. Dies ist vornehmlich auf die 1984 erfolgte Entdeckung einer neuen Knoten-Invarianten durch Vaughan F.R. Jones zurückzuführen. Die Theorie hat seither eine grosse Anzahl von neuen Anwendungen innerhalb und ausserhalb der Mathematik gefunden. Das oben erwähnte Problem allerdings ist trotz diesen Fortschritten immer noch offen, auch wenn man seiner Lösung nun sehr nahe zu sein scheint. *ust*

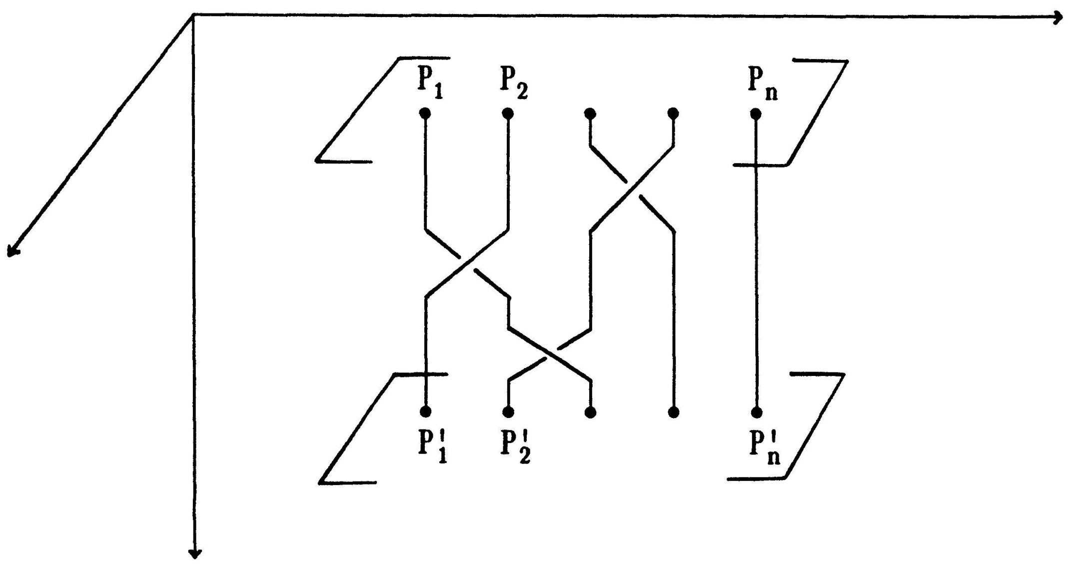

Fig. 1 A geometric braid

matical objects they were introduced by the German mathematician Emil Artin (1898–1962) in a paper from 1925, although the idea was already implicit in a paper of Hurwitz from 1891.

I shall give here a short introduction to the theory of braids, since these geometrical objects have recently been of central importance in several major developments in mathematics. Braids appear in many mathematical subjects: Topology (links, covering spaces, fixed point theory), Operator Algebras and Dynamical Systems (knotted orbits) to mention a few examples. There are known applications in Physics (topological quantum numbers, statistical mechanics), Chemistry (benzene rings) and Biology (DNA-molecules).

In the third paper of this series, "The Dirac String Problem" [El. Math. 49 (4)] I shall give an application of braids in connection with a problem from Physics, but here I shall concentrate on the algebraic structure connected with these geometrical objects.

Basically, a braid is a system of intertwining strings. Two braids with the same number of strings can be combined to form another braid by attaching one of the ends of the first braid to one of the ends of the second braid. Thereby one can do calculations with braids in much the same way as with positive real numbers being multiplied together. We shall now make all this precise.

Consider two fixed horizontal planes in 3-space. We think of these planes respectively as the *upper* and the *lower* plane. Mark n different points $P_1, \ldots, P_n$ on a line in the upper plane and project them orthogonally onto the lower plane to the points $P_1', \ldots, P_n'$; cf. Figure 1. Furthermore, let τ be a permutation of the numbers $\{1, \ldots, n\}$.

A (geometric) *braid* β on n *strings* and with *permutation* τ is a system of n strings in the space between the upper and the lower plane that connects the points P_i in the upper plane with the points $P'_{\tau(i)}$ in the lower plane, such that:

(i) Each string intersects each of the intermediate parallel planes between the upper and the lower plane exactly once.

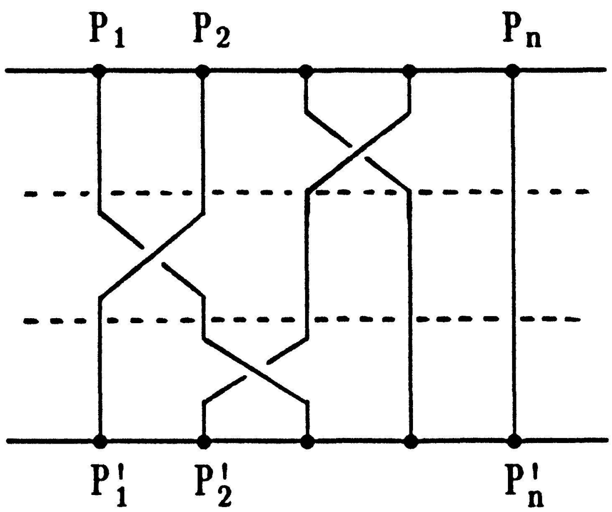

Fig. 2 Projection of braid

(ii) The n strings intersect each intermediate parallel plane between the upper and the lower plane in exactly n different points.

We shall also refer to β as an *n-braid.* As indicated in Figure 1, we think of a braid as hanging downwards.

To get a useful mathematical notion we have to define a notion of equivalence of braids. There are several ways in which to express the equivalence. In the definition that appeals most to the intuitive ideas, two braids on n strings are said to be *equivalent*, or to be the *same* braid, if they can be deformed into one another by a continuous deformation in 3-space keeping the regions above the upper plane and below the lower plane pointwise fixed. Henceforth, we shall not distinguish between the equivalence class of a braid and the braid itself.

After an arbitrarily small deformation, we can (and do) assume that a braid β consists of polygonal strings, and that the orthogonal projections of the strings onto the plane in 3-space that contains the endpoints $P_1, \ldots, P_n, P'_1, \ldots, P'_n$ of the braid have transversal crossings. By this projection we get a standard picture of the braid β as shown in Figure 2. We also remark that (up to equivalence) we can assume that crossings of strings occur at different levels; over- and undercrossings must be indicated.

In Figure 2 we have indicated how a braid can be resolved into elementary braids in which all strings, except for a neighbouring pair of strings, go right through from the upper to the lower plane, and the neighbouring pair of strings crosses each other exactly once.

For $1 \leq i \leq n-1$, we denote by σ_i the *elementary* geometric *n-braid* in which the ith string overcrosses the $(i+1)$th string exactly once and all other strings go right through from the upper to the lower plane. See Figure 3.

Let $B(n)$ denote the set of all equivalence classes of geometric n-braids. As we shall show now this set can be equipped with a product operation which gives it the mathematical structure of a group. A well known example of a group is the set of positive real numbers equipped with the usual product.

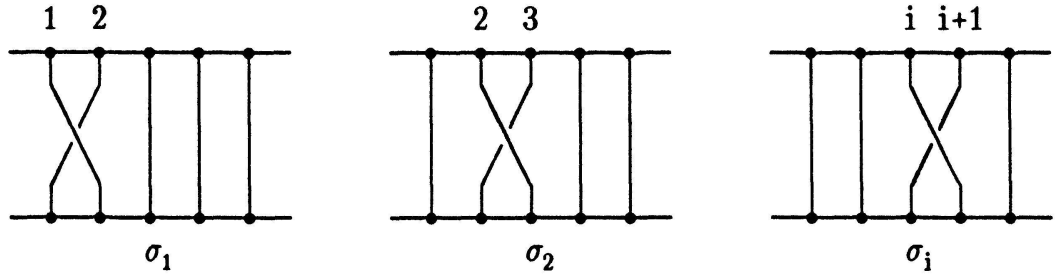

Fig. 3 Elementary braids

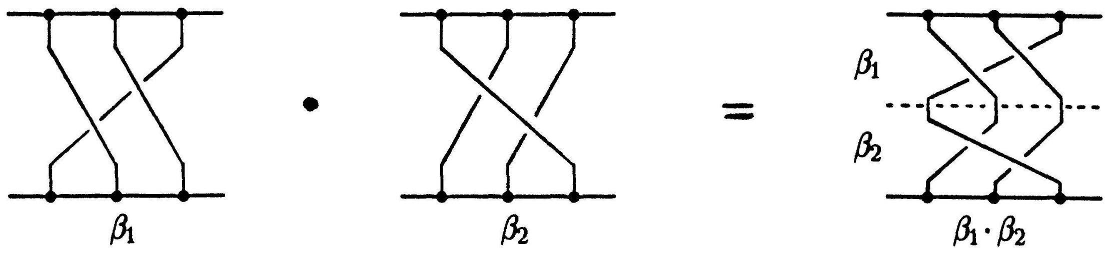

Fig. 4 Product of braids

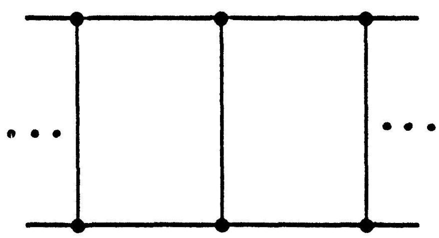

Fig. 5

Let β_1 and β_2 be geometric n-braids. Then we define the *product* of β_1 and β_2, denoted $\beta_1 \cdot \beta_2$, in the following way: First hang the braid β_2 under the braid β_1 by attaching the lower plane of β_1 to the upper plane of β_2. Then remove the plane along which the braids β_1 and β_2 are attached to each other. Now squeeze the resulting system of strings to lie between the upper and the lower plane, and we have the braid $\beta_1 \cdot \beta_2$. See Figure 4.

If we replace the braids β_1 and β_2 by equivalent braids β_1' and β_2', then it is clear that the product braids $\beta_1 \cdot \beta_2$ and $\beta_1' \cdot \beta_2'$ are equivalent, since deformations of β_1 to β_1' and β_2 to β_2' can be made in the respective layers before we squeeze and form the product. The product of braids is therefore well defined on equivalence classes of n-braids and thus induces a product in $B(n)$.

The *trivial* n-braid ϵ is the n-braid in which all strings go right through from the upper to the lower plane. In Figure 5 we show the projection of ϵ. It is clear, that ϵ is a *neutral element* for the product in $B(n)$, i.e. the product braids $\beta \cdot \epsilon$ and $\epsilon \cdot \beta$ are equivalent to β for every n-braid β.

We define the *inverse braid* β^{-1} of the braid β as the mirror image of β with respect to a horizontal plane between the upper and the lower plane. The projections of β and β^{-1} are shown in Figure 6.

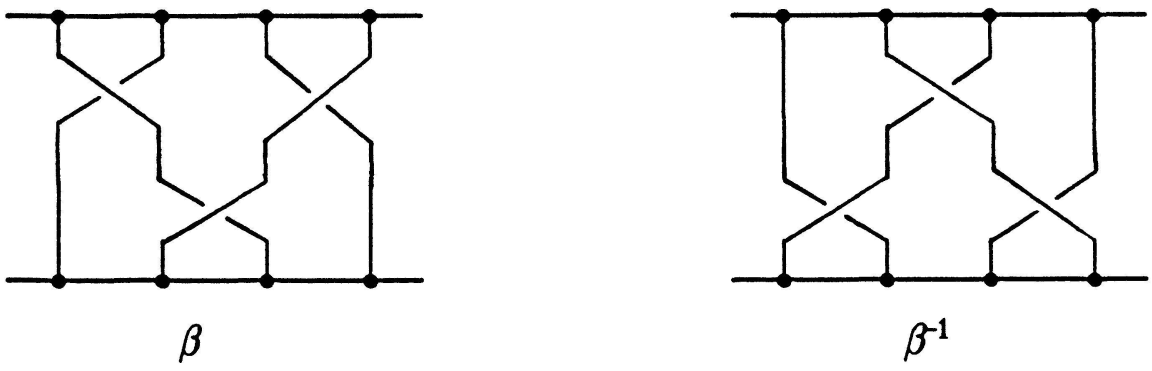

Fig. 6

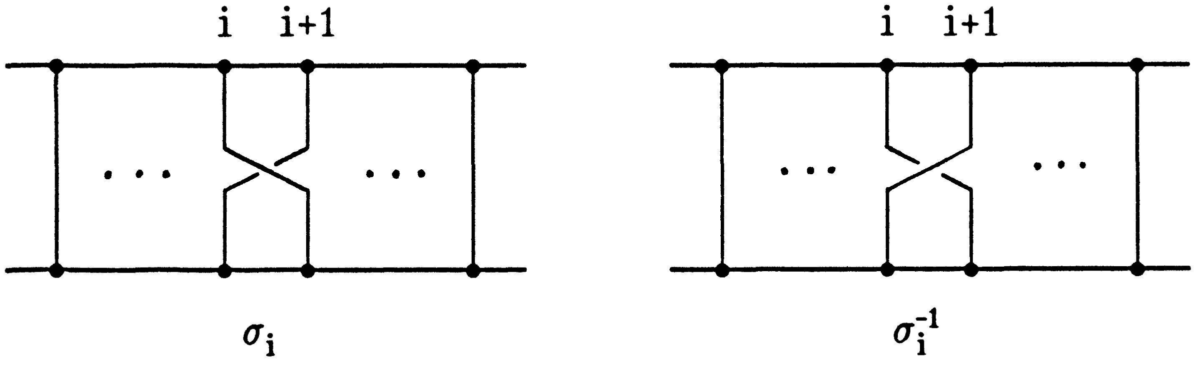

Fig. 7

It is not difficult to see that the equivalence class of β^{-1} is well defined from the equivalence class of β, and that the product braids $\beta \cdot \beta^{-1}$ and $\beta^{-1} \cdot \beta$ are equivalent to the trivial braid ϵ. The equivalence class of β^{-1} is therefore the inverse element to the equivalence class of β in $B(n)$.

For the elementary n-braid σ_i, $1 \leq i \leq n-1$, we get the inverse braid σ_i^{-1} by substituting in the standard projection the overcrossing of the $(i+1)$th string by the ith string with an undercrossing. See Figure 7.

It is not difficult to prove that if $B(n)$ is equipped with the above product we get a group with the trivial n-braid as neutral element and inverse elements as mentioned. This group is called the *(Artin) braid group.*

As it has already been indicated in Figure 2, it is intuitivly clear that the equivalence class of an arbitrary n-braid can be written as a product of the elementary n-braids σ_i, $1 \leq i \leq n-1$, and their inverse elements. From a group theoretical point of view, this means that the elementary n-braids $\sigma_1, \ldots, \sigma_{n-1}$ *generate* the group $B(n)$.

There are also group theoretical *relations* among the elements in $B(n)$. First we remark that the following relation holds:

(1) $\quad \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i \quad \text{for} \quad |i-j| \geq 2, \; 1 \leq i, j \leq n-1.$

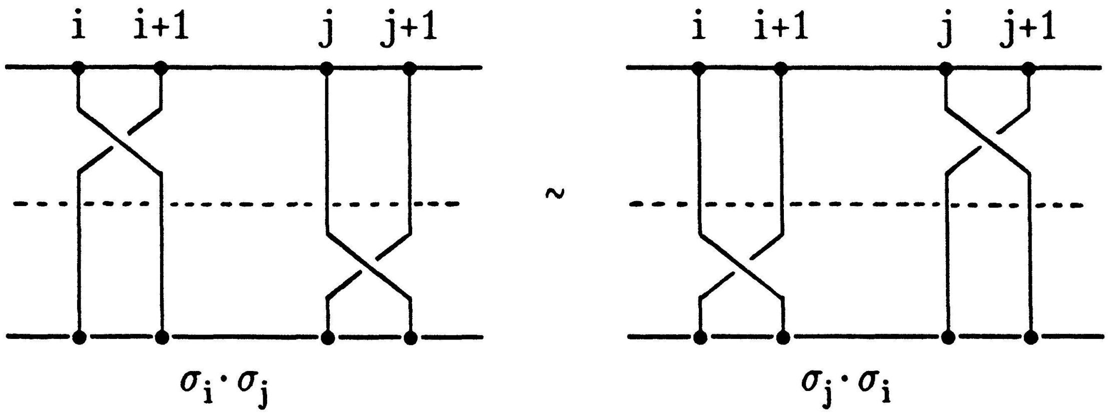

Fig. 8

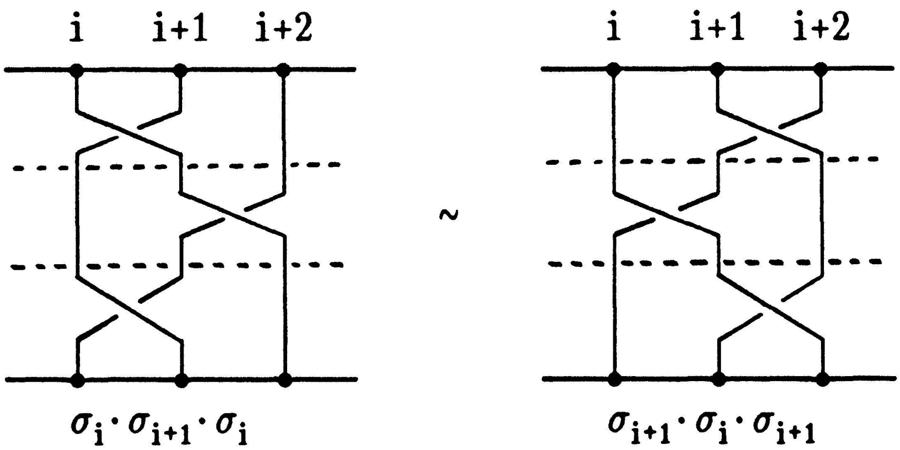

Fig. 9

The relation follows immediately, since the pair consisting of the ith and the $(i+1)$th string does not interfere with the pair consisting of the jth and the $(j+1)$th string with the given restrictions on i and j. In Figure 8 we illustrate the relation.

As illustrated in Figure 9, we also have the following relation in $B(n)$:

(2) $\quad \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} \quad \text{for} \quad 1 \leq i \leq n-2.$

Already in Artin's first paper on braid groups from 1925 a proof was given that an arbitrary relation among the elements in $B(n)$ can be deduced from relations of the types (1) and (2). This is actually quite technical to prove and we cannot indicate a proof here.

In group theory one says that the braid group $B(n)$ has a *presentation* with *generators* $\sigma_1, \ldots, \sigma_{n-1}$ and *generating relations*

(1) $\quad \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i \quad \text{for} \quad |i-j| \geq 2, \ 1 \leq i, j \leq n-1$

(2) $\quad \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} \quad \text{for} \ 1 \leq i \leq n-2.$

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