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The Magic World of Geometry — III. The Dirac String Problem

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The forms and relations in geometry are often hidden behind the phenomena they describe, so that mathematics as the explanation of the phenomena appears to be like a sixth sense in human beings. In the final article of this series of three articles with common subtitle "The Magic World of Geometry", I shall illustrate this by using the idea of braids to explain what is known as the Dirac string problem.

During his work in the 1920's to establish relativistic quantum mechanical models for the elementary particles, the Nobel laureate in Physics P.A.M. Dirac (1902–1984) found that he needed a notion of *spin*. Dirac predicted that an elementary particle with half-

Man stelle ein volles Glas Wasser auf die Handfläche, drehe die Hand, Handfläche nach oben, um eine volle Drehung gegen den Körper, bis sich die Hand und das darauf stehende Glas wieder vor dem Körper befinden, jetzt natürlich bei verdrehtem Arm. Nach einer weiteren vollen Drehung in derselben Richtung, diesmal über dem Kopf ausgeführt, ist die Verdrehung des Armes rückgängig gemacht. Während der ganzen Bewegung hat das Glas — hoffentlich — seine vertikale Position beibehalten. — In seinem dritten Beitrag in der Reihe *The magic world of geomery* zeigt uns V.L. Hansen, wie diese im Grunde einfache Bewegung eine Vielzahl von Bezügen innerhalb und ausserhalb der Mathematik aufweist: das Möbiusband, der quantenmechanische Spin von Elementarteilchen, das Diracsche "Stringproblem", die Lösung gewisser Puzzles, die Zopftheorie Artins, gewisse Eigenschaften der Gruppe der Rotationen im dreidimensionalen Raum, etc. — Wer möchte da nicht von Magie sprechen? *ust*

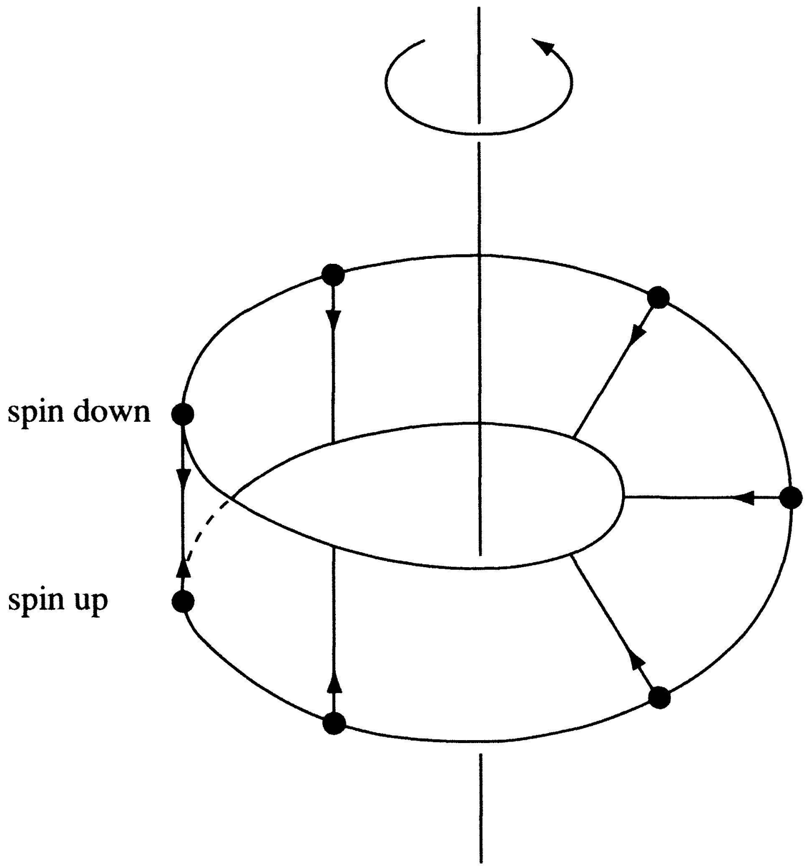

Fig. 1

spin, a so-called fermion (the electron, the proton and the neutron are examples), should have the following property: If the particle is turned 2π around an axis, then it is not the same particle any more — it is in a different state. But if you further turn it another 2π around the axis, so that we altogether have made two full turns, then the particle is back to normal. For the neutron this was only verified experimentally as late as 1975.

Quantum mechanically, *spin* can be represented by a *spin vector*. In Figure 1 we illustrate an elementary particle with spin $1/2$. During the turning around an axis the spin vector for the particle follows the generating line on a Möbius band (a band with a half twist), such that the particle from its original state (spin up) goes into a different state (spin down) after a single full turn, and only returns back to the original state (spin up) after a double rotation.

It may at first seem to contradict our intuition about objects in space that a particle could be connected in this way to its surroundings in a nontrivial topological way. But as a matter of fact such effects also occur in the macroscopical world.

A simple illustration can be given as follows. Put your right hand forward with the palm of the hand upwards. Turn the hand 2π with the palm of the hand pointing upwards during all of the motion. The hand is now back in its original position, but its connection to the surrounding space, namely the arm, is twisted. After a further turn through 2π, during which the arm is taken over your head, the twisting is removed, and you are back to normal. This process is part of a Phillippino wine dance where a dancer performs these movements with both hands simultaneously and with a wine glass standing on each hand.

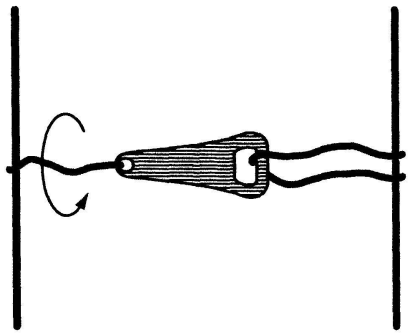

Fig. 2 No twisting of strings

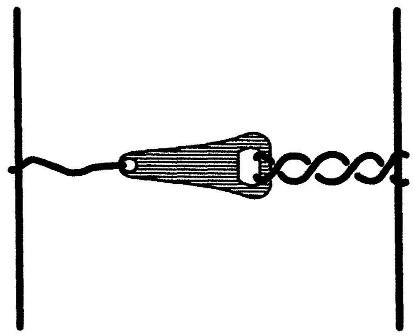

Fig. 3 Double twisting of strings

A more subtle demonstration was invented by Dirac himself and presented at lectures in the beginning of the 1930's. Take a solid object in 3-space (e.g. a wrench, or a bottle opener as we do here) and attach it to two posts (e.g. two table legs) by loose (or elastic) strings, say with one string from one end of the object and two strings from the other end of the object to the two posts respectively; see Figure 2. (Dirac himself used a pair of scissors as his object.) If you turn the object 2π around the "string-axis", the two strings going out to one of the posts will be twisted together. If you keep the object fixed, it is hardly a surprise that it is impossible to remove the twisting of the strings by passing them over and round the object. However, if you turn the object another 2π around the "string-axis", so that you have now altogether made two full turns, the twisting becomes more complicated (Figure 3) — but only apparently — for surprisingly enough, it is now possible to remove the twisting of the strings by passing them over and round the object.

It takes some interesting mathematics to explain the above phenomenon, and it is finding this explanation, which is known as *the Dirac string problem.* The problem was solved in 1942 by the English mathematician M.H.A. Newman using the mathematical theory of braids developed by Artin in the 1920's. There was however a small formal gap in Newman's proof, since he made use both of an algebraic and a geometric description of braids on a sphere, and these descriptions were not formally known to be equivalent at that time. The equivalence of the two descriptions of braids was first established by the American mathematicians E. Fadell and J. Van Buskirk in 1962.

In the series of pictures in Figure 4 it is shown how to remove the double twisting of the strings by taking the strings over and round the bottle opener.

An application which exploits the principle that after two full turns you are back to the initial position was patented in 1971 by the American D.A. Adams, who offered an ingenious solution to the problem of transferring electrical current to a rotating plate without the wires being entangled and breaking.

We shall now briefly present the mathematical clarification of the Dirac string problem.

Since we need to take the strings around the object (the bottle opener) we have to consider braids between two concentric spheres as in Figure 5. Such braids are defined in complete analogy to the definition of braids in the space between two fixed planes, just by substituting the two fixed horizontal planes and their intermediate planes by two fixed concentric spheres and their intermediate spheres. The set of equivalence classes

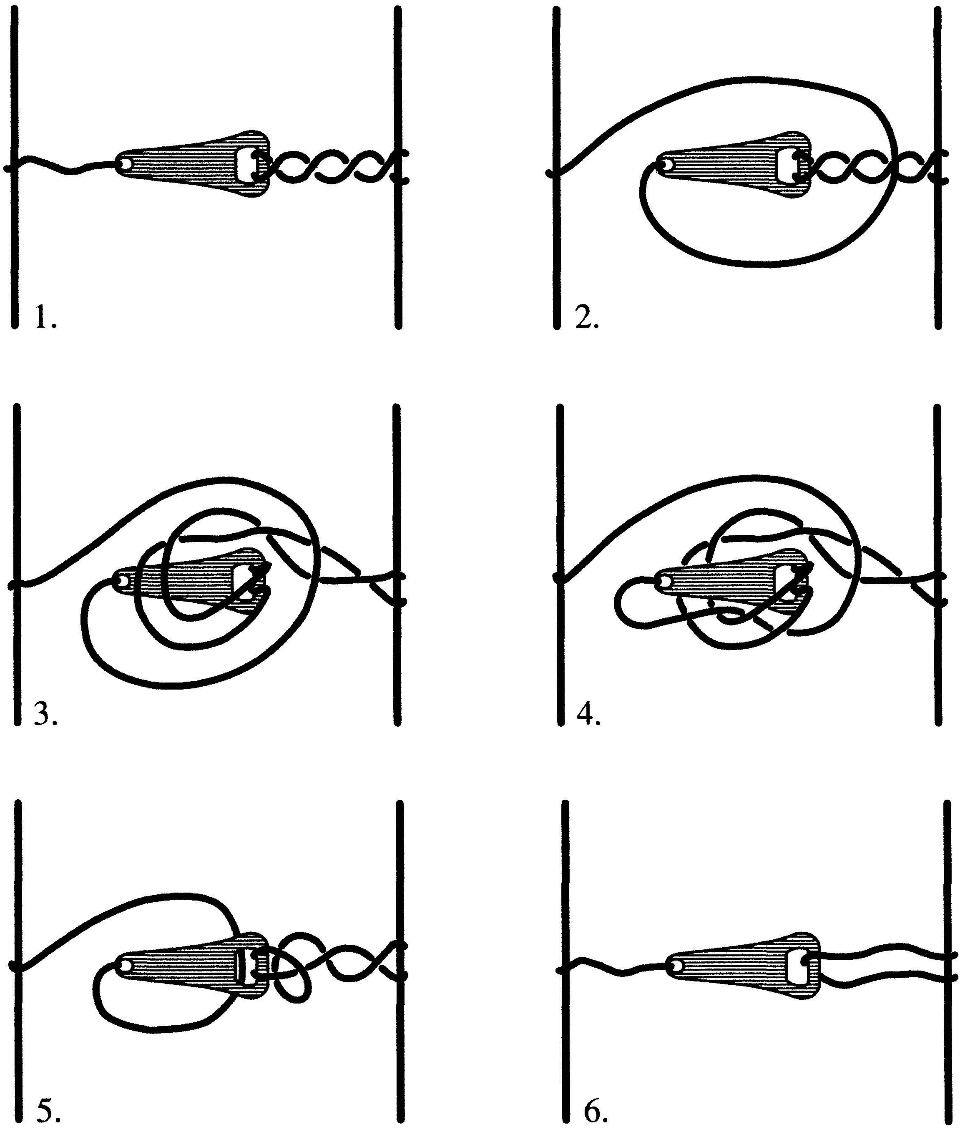

Fig. 4

of braids with n strings between spheres can likewise be equipped with a product and a group structure.

Now imagine that the interior sphere is free to rotate about its centre while the exterior sphere is kept fixed. Then we get a mathematical model of the Dirac string problem by letting the interior sphere model the bottle opener and the exterior sphere model the fixed surroundings to which we attach the strings. In Figure 6 we have the initial position, namely the trivial braid ϵ with 3 strings between the two spheres.

The braid Δ in Figure 7 represents the situation after one full turn of the bottle opener. For obvious reasons we call Δ the *Dirac braid.* After two full turns we get the braid in Figure 8. It is clear that this braid must be the product braid Δ^2.

The explanation of the Dirac string problem now boils down to the fact that Δ is an element of order 2 in the braid group on the sphere. In other words: *The Dirac braid Δ is different from the trivial braid ϵ, whereas the product braid Δ^2 is equivalent to ϵ.*

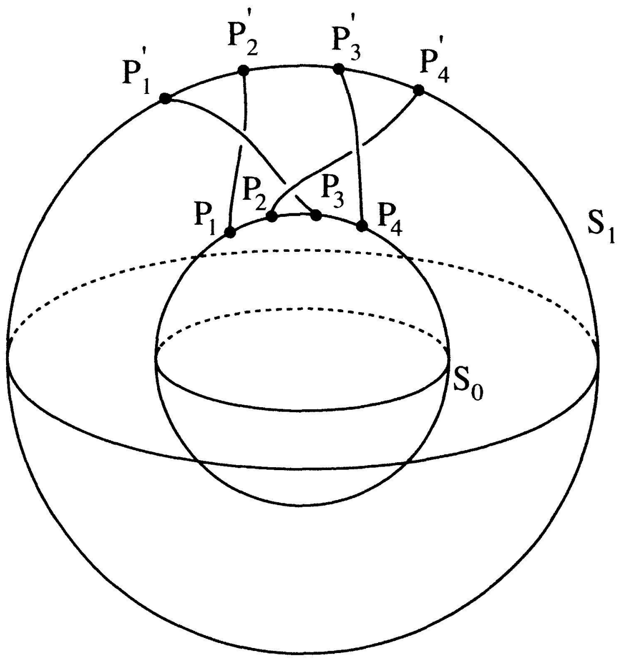

Fig. 5

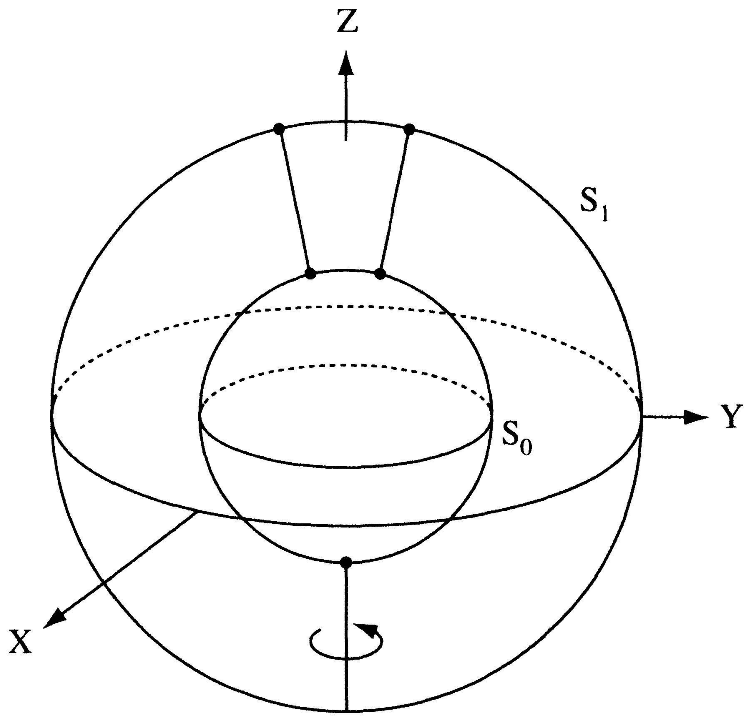

Fig. 6

The proof of this result uses well known topological properties of the group of rotations in 3-space. We shall outline the argument. First we remark that a full right-handed twist around a given oriented axis can be deformed into a full right-handed twist around any other arbitrarily chosen oriented axis. Two full right-handed twists around an oriented axis can therefore be deformed into the product of one full right-handed twist around

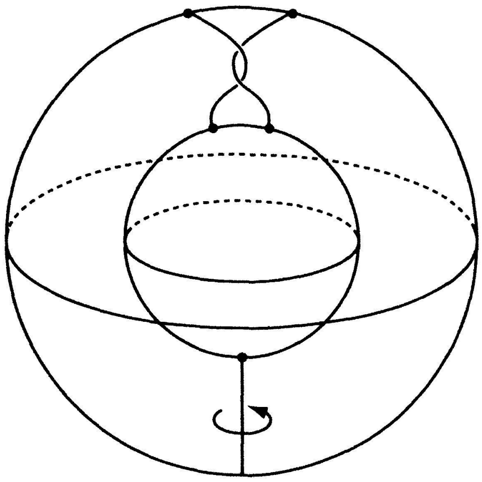

Fig. 7 The Dirac braid Δ

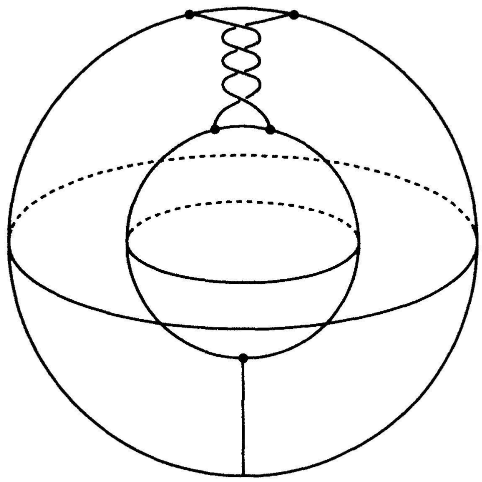

Fig. 8 The braid Δ^2

the given axis followed by one full right-handed twist around the axis with the opposite orientation. The last rotation, however, is clearly equivalent to one full left-handed twist around the originally given oriented axis. One full right-handed twist followed by one full left-handed twist around the same oriented axis can however be deformed into the trivial rotation. You can construct a deformation by starting the back-twisting in the left-handed direction at a continuously decreasing angle to the right-handed twist. This explains why a double twisting of the strings can be removed. A single twist of the strings cannot be removed since one full right-handed twist around an oriented axis can never be deformed into the trivial rotation.

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