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Covering a Square by Equal Circles

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One of the classical problems of discrete geometry is the following [2]: How must n equal non-overlapping circles be packed in the unit square so that the diameter d_n of the circles will be as large as possible? This *packing* problem has a dual counterpart in *covering*: How must the unit square be covered by n equal circles, so that the radius r_n of the circles will be as small as possible?

Results for the *packing* problem have been established for a relatively large number of values of n, see for example the surveys given in [1, 4, 5]. In contrast, surprisingly, no results seem to be known for the *covering* problem. Fejes Tóth's book [2] and even the

Wie müssen n Punkte in einem Quadrat verteilt werden, damit jeder Punkt des Quadrates möglichst nahe bei einem dieser n Punkte liegt? Diese Aufgabe kann als Frage nach einer Kreisüberdeckung des Quadrates formuliert werden: Wie überdeckt man auf optimale Art ein Quadrat durch n gleiche Kreisscheiben? Die Autoren des vorliegenden Beitrages gehen interessanterweise diese geometrische Frage nicht direkt an. Sie ordnen einer Überdeckung des Quadrates durch Kreisscheiben zuerst einen Graphen zu und diesem Graphen seinerseits ein physikalisches Modell bestehend aus starren Balken. Physikalische Überlegungen in diesem Modell ermöglichen schliesslich eine computerunterstützte numerische Suche nach den lokalen Optima. *ust*

recent surveys by Croft, Falconer and Guy [1] and by Moser and Pach [4] only mention an asymptotic analysis [7], and no actual circle arrangements.

The *density* D_n of a covering is defined as the ratio of the total area of the circles to the area of the square; for the unit square one has $D_n = nr_n^2\pi$. For solutions of the covering problem the density is a minimum. Lower bounds for the minimum density can be given by the formula

$$D_n^{\text{low}} = \frac{n\pi}{\left[-\frac{1}{4} + \left\{\frac{1}{16} + \frac{3\sqrt{3}}{2}n\right\}^{1/2}\right]^2}, \qquad n \geq 2 \tag{1}$$

which is based on the results of Verblunsky [7]. Upper bounds can most appropriately be given by explicit constructions of coverings.

The aim of this paper is to construct locally optimal coverings with n circles for $1 \leq n \leq 10$. The resulting upper bounds on the minimum diameter of the circles give upper bounds on the minimum density of coverings.

Following a suggestion of Fejes Tóth [2] (see the description by Meschkowski [3]) we associate with a covering system of circles a *bipartite graph*. It contains two kinds of *vertices*. The vertices of the first kind are the centers of the circles; the vertices of the second kind are those points where the boundaries of three or more circles (or one side of the square and the boundaries of two or more circles; or two sides of the square and the boundary of one or more circles) intersect, and which are not interior points of any other circle. (In the figures, the vertices of the first kind are marked by small circles, the vertices of the second kind carry no special mark.) The *edges* of the graph are straight line segments, drawn from each vertex of the second kind to the centers of the circles which intersect at that vertex. Since all the circles are of equal size, all edges are of equal length.

The idea of finding a locally optimal arrangement of circles covering the square is based on a "cooling technique" and on the theory of rigidity of graphs as developed by the authors [6] for covering a sphere by equal circles. The calculations start with an arrangement for which the graph is not rigid, but allows a finite amount of motion. The length of all the edges of the graph is then decreased simultaneously and in the same proportion. For this we regard the graph of the covering as a network of bars: It is modelled as a structure consisting of straight bars and frictionless pin joints. Change in length of the bars is attributed to a change in temperature. Since all the bars are identical, they contract equally during a uniform decrease of temperature. At a certain temperature the graph tightens, becomes rigid, and gets into a state of self-stress. In the physical model it is clear that in the case of an optimal arrangement further cooling will produce at most a tensional force (and not a compressing force) in the bars of the structure. Thus if in the actual arrangement any bar turns out to be in compression, then that bar should be removed. For the structure modified in this way — note that it may happen that some additional bars will have to be included —, we repeat this process of cooling and then removing the compressed bars. We continue in this way until the cooling process ceases to produce compression in any of the bars of the construction. This then indicates that

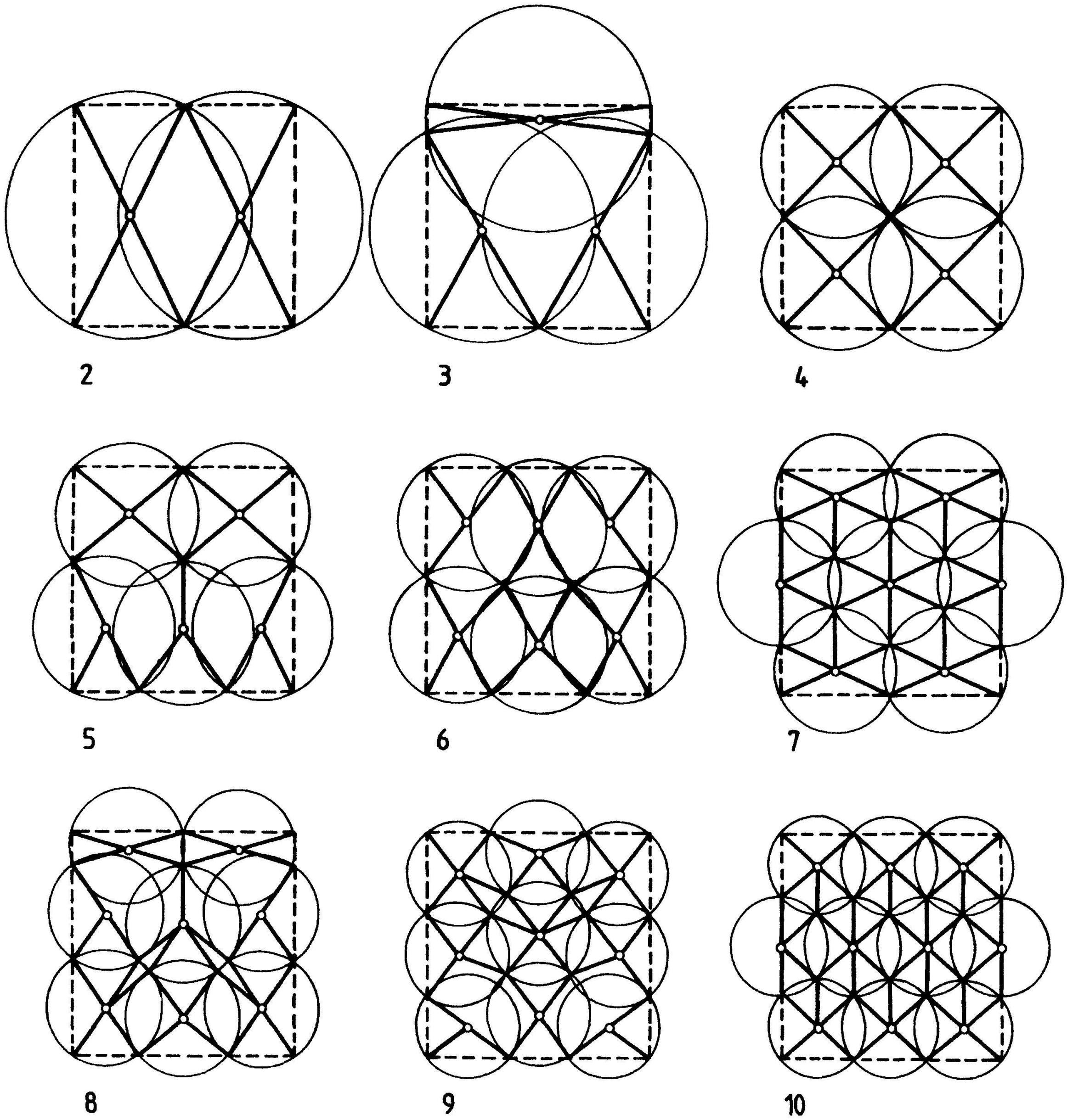

Fig. 1 Circle arrangement and the graph of the minimum covering of the square by n equal circles.

we have arrived at a situation where the length of the edges of the graph, that is, the radius of the circles, is a local minimum.

For actual computations we have used a version of the mathematical procedures applied in [6]. In the numerical investigations we have looked for the local minimum of the edge length in such a way that, in a state of self-stress, we have satisfied the equilibrium equations of the graph by iteration modifying the position of the vertices and gradually decreasing the error (the size of the unequilibrated force) at the vertices. The computations have been carried out by an IBM/AT 486 compatible personal computer. The FORTRAN programs used in the computations have been developed by us.

Applying this cooling technique we have discovered locally optimal coverings up to $n = 10$. Details of the calculations are omitted here, but the numerical data of the conjectured best coverings are collected in Table 1; the lower bounds for the extremal

n	Radius r_n	Density D_n	Lower bound of minimum density by formula (1)
1	0.7071067	1.5707963*	
2	0.5590169	1.9634954*	1.5051114
3	0.5038911	2.3930100*	1.4460203
4	0.3535533	1.5707963*	1.4118549
5	0.3261605	1.6710245	1.3889860
6	0.2989507	1.6846133	1.3723347
7	0.2742918	1.6545269	1.3595269
8	0.2605481	1.7061446	1.3492874
9	0.2306369	1.5040079	1.3408626
10	0.2182335	1.4962107	1.3337756

* It is easy to show that the density is a minimum.

Table 1 Conjectured best coverings of the unit square by n equal circles.

density have been calculated using formula (1). The circle configurations, together with the graphs of the coverings for $2 \leq n \leq 10$ are presented in Figure 1. For $n = 1$ the smallest circle covering a square is, of course, the circumcircle of the square.

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