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## Disjoint empty convex polygons in planar point sets

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Attila Gulyás was born in 1974 in Budapest. He studied mathematics at Eötvös Loránd University, Budapest and economics at Budapest University of Economic Sciences. Currently, he is a Ph.D. student in informatics at Eötvös Loránd University, Budapest. His research interests are combinatorics and insurance mathematics.

László Szabó was born in 1966 in Budapest. He studied mathematics at Eötvös Loránd University, Budapest, and received his Ph.D. from the Hungarian Academy of Sciences in 1996. His research interests are discrete geometry and convexity.

### 1 Introduction

Recently, the study of convex polygons has gained a renewed interest because of their importance in computer graphics, geometric learning theory, and artificial intelligence, for instance. Surprisingly, many simple questions are unanswered in this field. Let us start with a beautiful example. We say that a set of points in the plane is in general position if no three of the points lie on a line. Decades ago, Erdős, Klein, and Szekeres posed the problem of determining the maximum number  $f(k)$  of points in general position in the plane so that no  $k$  points form the vertex set of a convex polygon. Erdős and Szekeres [3] proved that

$$2^{k-2} \leq f(k) \leq \binom{2k-4}{k-2},$$

Fragen zu speziellen Konfigurationseigenschaften von Punkten in der Ebene sind seit jeher Gegenstand von Untersuchungen in der kombinatorischen Geometrie. Heutzutage kommt diesen Fragestellungen aufgrund der Anwendungsmöglichkeiten beim Design von Computergraphiken eine besondere Bedeutung zu. So stellt sich bei vorgelegter natürlicher Zahl  $k$  zum Beispiel die Frage nach der Maximalzahl von Punkten in allgemeiner Lage in der Ebene mit der Eigenschaft, dass in keinem Falle  $k$  dieser Punkte die Ecken eines konvexen Polygons bilden. Dieses Problem ist für  $k > 5$  bis heute ungelöst! In dem vorliegenden Beitrag von A. Gulyás und L. Szabó wird mit Hilfe eines raffinierten vollständigen Induktionsbeweises eine verwandte Problemstellung gelöst. *jk*

and conjectured that  $f(k)$  is equal to the lower bound. Surprisingly, this conjecture has been verified only for  $k = 3, 4, 5$ . Recently, the upper bound has been slightly improved by many authors, see [2, 6, 7]. The current record, due to Tóth and Valtr [7], is

$$f(k) \leq \binom{2k-5}{k-2} + 1.$$

Later, Erdős also posed the problem of determining the maximum number  $g(k)$  of points in general position in the plane so that no  $k$  points form the vertex set of an empty convex polygon, i.e., a convex polygon whose interior is disjoint from the point set. It is easy to see that  $g(3) = 2$  and  $g(4) = 4$ . Harborth [4] proved that  $g(5) = 9$ , and Horton [5] showed that  $g(k)$  is infinite for  $k \geq 7$ . It is a challenging open problem to decide whether  $g(6)$  is finite.

Let  $g_k(n)$  denote the minimum number of empty convex  $k$ -gons induced by the  $k$ -tuples of a set of  $n$  points in general position in the plane. Bárány and Füredi [1] proved that  $g_3(n) \geq n^2 - O(n \log n)$ ,  $g_4(n) \geq \frac{1}{4}n^2 - O(n)$ , and  $g_5(n) \geq \lfloor n/10 \rfloor$ . We note that the last bound can easily be improved to  $g_5(n) \geq \lfloor (n-4)/6 \rfloor$ . On the other hand, Valtr [8] showed that  $g_3(n) \leq 1.8n^2$ ,  $g_4(n) \leq 2.42n^2$ , and  $g_5(n) \leq 1.46n^2$ .

It is obvious that the  $k$ -tuples of a set of  $n$  points in general position in the plane always induce a family of  $\lfloor n/(g(k)+1) \rfloor$  disjoint empty convex  $k$ -gons, and this bound is tight for  $k = 3$ . In this paper we consider the case  $k = 4$  and we prove

**Theorem 1** *The quadruples of a set of  $n$  points in general position in the plane always induce a family of  $\lfloor 2n/9 \rfloor$  disjoint empty convex quadrangles.*

We also show that the bound  $\lfloor 2n/9 \rfloor$  cannot be improved for  $n \leq 21$ .

## 2 Proof of Theorem 1

First we prove that any set  $\mathcal{P}$  of nine points in general position in the plane contains two disjoint empty convex quadrangles. Let  $p_1, p_2, \dots, p_m$  denote the vertices of the convex hull of  $\mathcal{P}$  in a counterclockwise order (we will use the convention that  $p_i = p_j$  if  $i \equiv j \pmod{m}$ ). Observe that if  $\triangle p_{i-1}p_i p_{i+1}$  is an empty triangle of  $\mathcal{P}$  for some  $1 \leq i \leq m$ , then  $\mathcal{P}$  contains two disjoint empty quadrangles. Indeed, among  $\mathcal{P} \setminus \{p_{i-1}, p_i, p_{i+1}\}$  choose a point  $r$  whose distance from the line  $p_{i-1}p_{i+1}$  is minimal. Now  $p_{i-1}p_i p_{i+1}r$  is an empty convex quadrangle and the remaining five points of  $\mathcal{P}$  also contain an empty convex quadrangle which is obviously disjoint from  $p_{i-1}p_i p_{i+1}r$ . Therefore, in what follows, we will assume that  $\triangle p_{i-1}p_i p_{i+1}$  is not empty for  $1 \leq i \leq m$ . This immediately implies among others that  $m \leq 6$ .

**Case 1.**  $m = 6$ . Let  $q_1, q_2, q_3$  denote the points of  $\mathcal{P}$  lying in the interior of the convex hull of  $\mathcal{P}$ . Without loss of generality we may assume that  $q_i \in \triangle p_{2i-2}p_{2i-1}p_{2i} \cap \triangle p_{2i-1}p_{2i}p_{2i+1}$  for  $i = 1, 2, 3$  (do not forget that no  $\triangle p_{i-1}p_i p_{i+1}$  is empty,  $1 \leq i \leq 6$ ). Then  $p_1q_1q_3p_6$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by the line  $p_2p_5$ . Now we are done, since the set of the remaining five points necessarily contains an empty convex quadrangle, which is, of course, disjoint from  $p_1q_1q_3p_6$ .

**Case 2.**  $m = 5$ . Let  $q_1, q_2, q_3, q_4$  denote the points of  $\mathcal{P}$  lying in the interior of the convex hull of  $\mathcal{P}$ . A point of  $\{q_1, q_2, q_3, q_4\}$  will be called special if it is contained in  $\Delta p_{i-2}p_{i-1}p_i \cap \Delta p_{i-1}p_i p_{i+1}$  for some  $1 \leq i \leq 5$ . Obviously, at least one point of  $\{q_1, q_2, q_3, q_4\}$  is special.

**Case 2.1.** Exactly one point of  $\{q_1, q_2, q_3, q_4\}$ , say  $q_4$ , is special. Without loss of generality we may assume that  $q_4 \in \Delta p_1 p_4 p_5 \cap \Delta p_3 p_4 p_5$  and  $q_i \in \Delta p_{i-2} p_{i-1} p_i \cap \Delta p_{i-1} p_i p_{i+1}$  for  $i = 1, 2, 3$ . Now  $p_1 q_1 q_2 p_2$  and  $p_3 q_3 q_4 p_4$  are disjoint empty convex quadrangles (they are separated by the line joining  $p_5$  and  $\overline{p_1 p_3} \cap \overline{p_2 p_4}$ ).

**Case 2.2.** Exactly two points of  $\{q_1, q_2, q_3, q_4\}$ , say  $q_1, q_2$ , are special. Then  $\{q_1, q_2\} \subseteq \Delta p_{i-1} p_i p_{i+1}$  for at most one  $1 \leq i \leq 5$ .

**Case 2.2.1.** For some  $1 \leq i \leq 5$ , the set  $\{q_1, q_2\} \subseteq \Delta p_{i-1} p_i p_{i+1}$ . Without loss of generality we may assume that  $q_j \in \Delta p_{j-2} p_{j-1} p_j \cap \Delta p_{j-1} p_j p_{j+1}$  for  $j = 1, 2$  and  $q_j \in \Delta p_{j-2} p_j p_{j+2} \cap \Delta p_{j-1} p_j p_{j+1}$  for  $j = 3, 4$ . Now  $p_3 p_4 q_4 q_3$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_2 p_5$ .

**Case 2.2.2.** No  $\Delta p_{i-1} p_i p_{i+1}$  contains both  $q_1$  and  $q_2$ ,  $1 \leq i \leq 5$ . Without loss of generality we may assume that  $q_1 \in \Delta p_1 p_2 p_5 \cap \Delta p_1 p_2 p_3$ ,  $q_2 \in \Delta p_2 p_3 p_4 \cap \Delta p_3 p_4 p_5$ , and  $q_3 \in \Delta p_1 p_4 p_5 \cap \Delta p_2 p_3 p_5$ . Let  $u = p_2 q_3 \cap \overline{p_4 p_5}$  and  $v = p_3 q_3 \cap \overline{p_5 p_1}$  (see Figure 1).

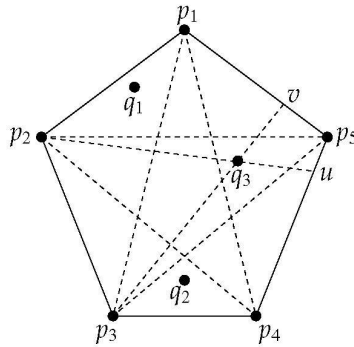


Fig. 1

If  $q_4$  is contained in the quadrangle  $p_2 p_3 p_4 u$ , then  $p_1 p_5 q_3 q_1$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_2 q_3$ . Similarly, if  $q_4$  is contained in the quadrangle  $p_1 p_2 p_3 v$ , then  $p_4 p_5 q_3 q_2$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_3 q_3$ . Finally, if  $q_4$  is contained in the quadrangle  $u p_5 v q_3$ , then  $p_2 p_3 q_2 q_1$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_1 p_4$ .

**Case 2.3.** Exactly three points of  $\{q_1, q_2, q_3, q_4\}$ , say  $q_1, q_2, q_3$ , are special.

**Case 2.3.1.** For some  $1 \leq i \leq 5$ ,  $\Delta p_{i-2}p_{i-1}p_i \cap \Delta p_{i-1}p_i p_{i+1}$  contains two points of  $\{q_1, q_2, q_3\}$ . Without loss of generality we may assume that  $\{q_1, q_2\} \subseteq \Delta p_1 p_4 p_5 \cap \Delta p_1 p_2 p_5$ ,  $q_3 \in \Delta p_1 p_2 p_3 \cap \Delta p_2 p_3 p_4$ , and  $q_4 \in \Delta p_2 p_4 p_1 \cap \Delta p_3 p_4 p_5$ . Now  $p_3 p_4 q_4 q_3$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_2 p_5$ .

**Case 2.3.2.** No  $\Delta p_{i-2}p_{i-1}p_i \cap \Delta p_{i-1}p_i p_{i+1}$  contains two points of  $\{q_1, q_2, q_3\}$ ,  $1 \leq i \leq 5$ . Then, among the triangles  $\Delta p_{i-1}p_i p_{i+1}$ ,  $1 \leq i \leq 5$ , one or two contain two points of  $\{q_1, q_2, q_3\}$ .

**Case 2.3.2.1.** Among the triangles  $\Delta p_{i-1}p_i p_{i+1}$ ,  $1 \leq i \leq 5$ , exactly one contains two points of  $\{q_1, q_2, q_3\}$ . Without loss of generality we may assume that  $q_j \in \Delta p_{j-1}p_j p_{j+1} \cap \Delta p_j p_{j+1} p_{j+2}$  for  $j = 1, 2$ ,  $q_3 \in \Delta p_3 p_4 p_5 \cap \Delta p_4 p_5 p_1$ , and  $q_4$  is not separated from  $q_2$  by the line  $p_2 q_3$ . Now  $p_1 p_5 q_3 q_1$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_2 q_3$ .

**Case 2.3.2.2.** Among the triangles  $\Delta p_{i-1}p_i p_{i+1}$ ,  $1 \leq i \leq 5$ , exactly two contain two points of  $\{q_1, q_2, q_3\}$ . Without loss of generality we may assume that  $q_j \in \Delta p_{j-1}p_j p_{j+1} \cap \Delta p_j p_{j+1} p_{j+2}$  for  $j = 1, 2, 3$ , and  $q_4 \in \Delta p_2 p_3 p_5 \cap \Delta p_1 p_4 p_5$ . Now  $p_1 p_5 q_4 q_1$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_2 p_4$ .

**Case 2.4.** All four points of  $\{q_1, q_2, q_3, q_4\}$  are special. Then there are three points of  $\{q_1, q_2, q_3, q_4\}$ , say  $q_1, q_2, q_3$ , so that no  $\Delta p_{i-2}p_{i-1}p_i \cap \Delta p_{i-1}p_i p_{i+1}$  contains more than one point of  $\{q_1, q_2, q_3\}$ ,  $1 \leq i \leq 5$ . Without loss of generality we may assume that  $q_j \in \Delta p_{j-1}p_j p_{j+1} \cap \Delta p_j p_{j+1} p_{j+2}$  for  $j = 1, 2$ .

**Case 2.4.1.** The point  $q_3$  is in  $\Delta p_3 p_4 p_5 \cap \Delta p_4 p_5 p_1$ . Without loss of generality we may assume that  $q_4$  is not separated from  $q_2$  by the line  $p_2 q_3$ . Now  $p_1 p_5 q_3 q_1$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_2 q_3$ .

**Case 2.4.2.** The point  $q_3$  is contained in  $\Delta p_2 p_3 p_4 \cap \Delta p_3 p_4 p_5$  or  $\Delta p_4 p_5 p_1 \cap \Delta p_5 p_1 p_2$ , say in  $\Delta p_2 p_3 p_4 \cap \Delta p_3 p_4 p_5$ . Then  $q_4$  is contained in  $\Delta p_3 p_4 p_5 \cap \Delta p_4 p_5 p_1$  or  $\Delta p_4 p_5 p_1 \cap \Delta p_5 p_1 p_2$ , say in  $\Delta p_3 p_4 p_5 \cap \Delta p_4 p_5 p_1$  (do not forget that  $\Delta p_1 p_4 p_5$  is not empty). Now  $p_1 p_5 q_4 q_1$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_2 p_4$ .

**Case 3.**  $m = 4$ . Let  $q_1, q_2, q_3, q_4, q_5$  denote the points of  $\mathcal{P}$  lying in the interior of the convex hull of  $\mathcal{P}$ . Let  $u = \overline{p_1 p_3} \cap \overline{p_2 p_4}$ .

**Case 3.1.** No  $q_1, q_2, q_3, q_4, q_5$  is contained in  $\Delta p_i u p_{i+1}$  for some  $1 \leq i \leq 4$ , say in  $\Delta p_1 u p_2$ . Without loss of generality we may assume that  $\angle q_1 p_1 p_2 < \angle q_j p_1 p_2$  for  $2 \leq j \leq 5$ , and  $\angle p_1 q_1 q_2 < \angle p_1 q_1 q_k$  for  $3 \leq k \leq 5$ . Now  $q_1 \in \Delta p_2 u p_3$  and the line  $q_1 q_2$  intersects  $\overline{p_1 p_4}$  since  $\Delta p_1 p_2 p_3$  and  $\Delta p_1 p_2 p_4$  are not empty. Then  $p_1 p_2 q_1 q_2$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $q_1 q_2$ .

**Case 3.2.** All  $\Delta p_i u p_{i+1}$  contain at least one point of  $\{q_1, q_2, q_3, q_4, q_5\}$ ,  $1 \leq i \leq 4$ . Without loss of generality we may assume that  $q_i \in \Delta p_i u p_{i+1}$  for  $1 \leq i \leq 4$  and  $q_5 \in \Delta p_4 u p_1$  (see Figure 2).

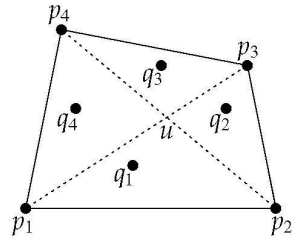


Fig. 2

**Case 3.2.1.** The line  $q_4q_5$  does not intersect  $\overline{p_1p_4}$ . If  $q_4q_5$  separates both  $q_1$  and  $q_3$  from  $p_1$  and  $p_4$ , then  $p_1p_4q_4q_5$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $q_4q_5$ . If  $q_4q_5$  does not separate  $q_3$  and  $q_1$  from  $p_1$  and  $p_4$ , then  $p_1p_4q_1q_3$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $q_4q_5$ . Finally, if exactly one point of  $\{q_1, q_3\}$ , say  $q_1$ , is separated from  $p_1$  and  $p_4$  by  $q_4q_5$ , then  $p_1p_4q_4q_3$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $q_4q_5$ .

**Case 3.2.2.** The line  $q_4q_5$  intersects  $\overline{p_1p_4}$ . Without loss of generality we may assume that  $q_4q_5$  is disjoint from  $\triangle p_1up_2$ . Now  $p_1q_1q_4q_5$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_2p_4$ .

**Case 4.**  $m = 3$ . Let  $q_1, q_2, q_3, q_4, q_5, q_6$  denote the points of  $\mathcal{P}$  lying in the interior of the convex hull of  $\mathcal{P}$ . Without loss of generality we may assume that  $\angle p_2p_1q_1 < \angle p_2p_1q_i$  for  $2 \leq i \leq 6$  and  $\angle p_1q_1q_2 < \angle p_1q_1q_i < \angle p_1q_1q_6$  for  $3 \leq i \leq 5$ . Let  $u = p_1q_1 \cap \overline{p_2p_3}$  and  $v = p_2q_1 \cap \overline{p_1p_3}$  (see Figure 3).

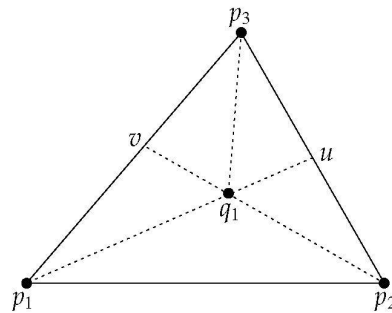


Fig. 3

It is obvious that  $\triangle p_1up_2$  is empty. If  $\triangle p_1q_1v$  is not empty, then  $p_1p_2q_1q_2$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $q_1q_2$ . Thus, in what follows, we will assume that  $\triangle p_1q_1v$  is empty. If  $\triangle q_1p_3v$  is empty, then  $p_1p_3q_2q_1$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $q_1q_2$ . Thus, in what follows, we will also assume that  $\triangle q_1p_3v$  is not empty. Similarly,

if  $\triangle q_1 p_3 u$  is empty, then  $p_2 p_3 q_6 q_1$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $q_1 q_6$ . Thus, in what follows, we will also assume that  $\triangle q_1 p_3 u$  is not empty. For technical reasons, in the remaining part of the proof we will disregard the special choice of  $q_2$  and  $q_6$ .

**Case 4.1.** Exactly one point of  $\{q_2, q_3, q_4, q_5, q_6\}$  is contained in  $\triangle q_1 p_3 v$  or  $\triangle q_1 p_3 u$ . Without loss of generality we may assume that  $q_2$  is contained in  $\triangle q_1 p_3 u$  and  $q_3, q_4, q_5, q_6$  are contained in  $\triangle q_1 p_3 v$ .

**Case 4.1.1.** Not all  $q_3, q_4, q_5, q_6$  are separated from  $p_3$  by  $p_2 q_2$ . Without loss of generality we may assume that  $\angle p_3 q_2 q_3 < \angle p_3 q_2 q_i$  for  $4 \leq i \leq 6$ . Now  $p_2 p_3 q_3 q_2$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $q_2 q_3$ .

**Case 4.1.2.** All  $q_3, q_4, q_5, q_6$  are separated from  $p_3$  by  $p_2 q_2$ . Suppose that  $p_1 p_3$  and  $q_1 q_2$  are not parallel (the case where  $p_1 p_3$  and  $q_1 q_2$  are parallel can be settled similarly). Let  $w = p_1 p_3 \cap q_1 q_2$ . Without loss of generality we may assume that  $\angle q_1 w q_3 < \angle q_1 w q_i$  for  $4 \leq i \leq 6$  (see Figure 4).

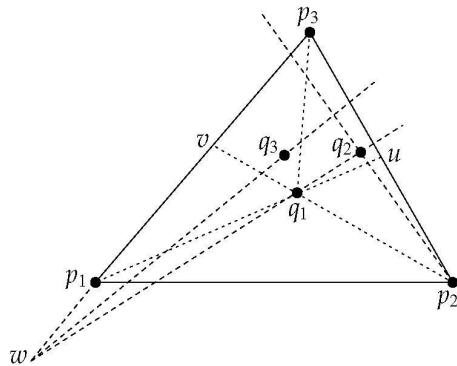


Fig. 4

Now  $p_2 q_1 q_3 q_2$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $w q_3$ .

**Case 4.2.** Exactly two points of  $\{q_2, q_3, q_4, q_5, q_6\}$  are contained in  $\triangle q_1 p_3 v$  or  $\triangle q_1 p_3 u$ . Without loss of generality we may assume that  $q_2, q_3$  are contained in  $\triangle q_1 p_3 u$  and  $q_4, q_5, q_6$  are contained in  $\triangle q_1 p_3 v$ .

**Case 4.2.1.** The line  $q_2 q_3$  does not intersect  $\overline{p_2 p_3}$ . Now  $p_2 p_3 q_3 q_2$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_3 q_1$ .

**Case 4.2.2.** The line  $q_2 q_3$  intersects  $\overline{p_2 p_3}$  and  $\overline{q_1 p_3}$ . Now  $p_2 q_3 q_2 q_1$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_3 q_1$ .

**Case 4.2.3.** The line  $q_2q_3$  intersects  $\overline{p_2p_3}$  and  $\overline{q_1p_2}$ . Without loss of generality we may assume that  $q_3$  separates  $q_2$  from  $q_2q_3 \cap \overline{p_2p_3}$  (see Figure 5). If  $p_1q_4q_5q_6$  is a convex quadrangle then we are done. Indeed, now  $p_1q_4q_5q_6$  is empty and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_3q_1$ . Thus, in what follows, we will assume, without loss of generality, that  $q_4$  is contained in  $\triangle p_1q_5q_6$ . We will also assume that  $q_6$  separates  $q_5$  from  $q_5q_6 \cap \overline{p_1p_3}$  if  $q_5q_6$  intersects  $\overline{p_1p_3}$  and that  $q_6$  separates  $q_5$  from  $q_5q_6 \cap \overline{p_3q_1}$  if  $q_5q_6$  does not intersect  $\overline{p_1p_3}$ .

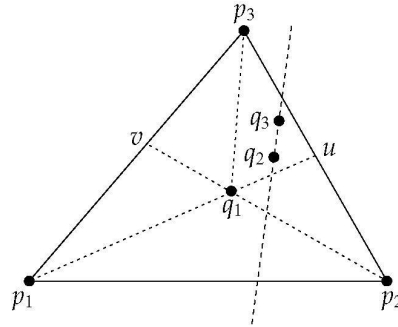


Fig. 5

**Case 4.2.3.1.** The line  $q_5q_6$  separates  $p_3$  from  $q_2$  and  $q_3$ . If  $q_5q_6$  does not intersect  $\overline{p_1p_3}$ , then  $p_1p_3q_6q_4$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $q_5q_6$ . On the other hand, if  $q_5q_6$  intersects  $\overline{p_1p_3}$ , then either  $p_1p_3q_6q_4$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $q_4q_6$  or  $p_3q_6q_4q_5$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_1q_5$ .

**Case 4.2.3.2.** The line  $q_5q_6$  separates  $q_2$  and  $q_3$ . If  $q_5q_6$  intersects  $\overline{p_1p_3}$ , then  $p_1q_1q_2q_4$  and  $p_3q_3q_5q_6$  are disjoint empty convex quadrangles. On the other hand, if  $q_5q_6$  does not intersect  $\overline{p_1p_3}$ , then  $p_3q_4q_6q_3$  and either  $p_1q_1q_2q_5$  or  $p_2q_1q_5q_2$  are disjoint empty convex quadrangles.

**Case 4.2.3.3.** The line  $q_5q_6$  does not separate  $p_3$  from  $q_2$  and  $q_3$ . If  $q_5q_6$  does not intersect  $\overline{p_1p_3}$ , then  $p_2q_1q_5q_6$  is an empty convex quadrangle and it is separated from the remaining five points of  $\mathcal{P}$  by  $q_5q_6$ . If  $q_5q_6$  intersects  $\overline{p_1p_3}$  and  $\overline{p_3q_1}$ , then  $p_3q_6q_2q_3$  and  $p_1q_4q_5q_1$  are disjoint empty convex quadrangles. Thus, we will assume that  $q_5q_6$  intersects  $\overline{p_1p_3}$  and  $\overline{p_1q_1}$ . If  $p_1p_3q_6q_4$  is a convex quadrangle, then we are done. Indeed, now  $p_1p_3q_6q_4$  is empty and it is separated from the remaining five points of  $\mathcal{P}$  by  $p_3q_6$ . Thus, we will assume that  $p_3q_6q_4q_5$  is an empty convex quadrangle. If  $p_3$  and  $q_3$  are separated by  $p_1q_5$ , then  $p_3q_6q_4q_5$  is separated from the remaining five points of  $\mathcal{P}$  by  $p_1q_5$ . On the other hand, if  $p_3$  and  $q_3$  are not separated by  $p_1q_5$ , then  $p_3q_6q_4q_3$  is an empty convex quadrangle, and either  $p_1q_1q_2q_5$  or  $p_2q_1q_5q_2$  is an empty convex quadrangle disjoint from  $p_3q_6q_4q_3$ .



Thus we have proved that any set  $\mathcal{P}$  of nine points in general position in the plane contains two disjoint empty convex quadrangles.

The proof of Theorem 1 will be done by induction on  $n$ . We know that the assertion is true for  $n \leq 9$ . Let  $n \geq 10$  and consider a set  $\mathcal{P}$  of  $n$  points in general position. It is obvious that there exists a line which cuts  $\mathcal{P}$  into two disjoint sets  $\mathcal{P}_1$  and  $\mathcal{P}_2$  of 9 and  $n - 9$  points, respectively. Then, by the induction hypothesis,  $\mathcal{P}_1$  contains two disjoint empty convex quadrangles and  $\mathcal{P}_2$  contains  $\lfloor 2(n - 9)/9 \rfloor = \lfloor 2n/9 \rfloor - 2$  disjoint empty convex quadrangles. Thus  $\mathcal{P}$  contains  $\lfloor 2n/9 \rfloor$  disjoint empty convex quadrangles.

### 3 Constructions

It is easy to find a set of eight points in general position in the plane which does not contain two disjoint empty convex quadrangles. Indeed, if  $p_1, p_2, p_3, p_4$  are the vertices of a square in a counterclockwise order and  $q_i$  is an interior point of  $p_1p_2p_3p_4$  sufficiently close to the midpoint of  $\overline{p_i p_{i+1}}$ ,  $1 \leq i \leq 4$ , then  $\{p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4\}$  is just such a point set.

Next we show that, for each  $m \geq 3$ , there exists a set of  $n = 4m + 1$  points in general position which does not contain  $m$  disjoint empty convex quadrangles.

Let  $p_1, p_2, \dots, p_{2m}$  be the vertices of a regular  $2m$ -gon  $C$  in a counterclockwise order and let  $q_i$  be an interior point of  $C$  sufficiently close to the midpoint of  $\overline{p_i p_{i+1}}$ ,  $1 \leq i \leq 2m$ . Furthermore, let  $r$  be a point sufficiently close to the centre of  $C$  so that  $\mathcal{P} = \{p_1, \dots, p_{2m}, q_1, \dots, q_{2m}, r\}$  is in general position. Suppose, for contradiction, that  $\mathcal{P}$  contains  $m$  disjoint empty convex quadrangles  $Q_1, Q_2, \dots, Q_m$ .

Let  $p_{i_1}$  and  $p_{i_2}$  two vertices of  $C$  so that they belong to  $Q_i$  for some  $1 \leq i \leq m$  and the number  $l$  of vertices of  $C$  on the shorter arc of  $C$  bounded by  $p_{i_1}$  and  $p_{i_2}$  is as small as possible. Now, a very simple counting argument shows that  $l \leq 4$ . Thus, without loss of generality we may assume that  $i_1 = 1$  and  $i_2 \in \{2, 3, 4\}$ . If  $i_2 = 2$ , then  $q_1$  cannot be a vertex of a quadrangle. If  $i_2 = 3$ , then  $p_2$  cannot be a vertex of a quadrangle. Finally, if  $i_2 = 4$ , then  $p_2$  or  $p_3$  cannot be a vertex of a quadrangle. Next, let  $p_{j_1}$  and  $p_{j_2}$  be two vertices of the longer arc of  $C$  bounded by  $p_{i_1}$  and  $p_{i_2}$  so that they belong to  $Q_j$  for some  $1 \leq j \leq m$  and the number  $l'$  of vertices of  $C$  on the arc of  $C$  bounded by  $p_{j_1}$  and  $p_{j_2}$  is as small as possible. Again, a very simple counting argument shows that  $l' \leq 4$  and, similarly as before, we can find a point of  $\mathcal{P}$  different from  $q_1, p_2, p_3$  which cannot be a vertex of a quadrangle. Thus  $\mathcal{P}$  necessarily contains two points which are not vertices of the quadrangles  $Q_1, Q_2, \dots, Q_{2m}$ , a contradiction.

Note that the above constructions show that the bound in Theorem 1 is tight for  $n \leq 21$ .

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