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## On summing to arbitrary real numbers

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Following Erdős [1] we say a sequence  $\{a_n\}_{n=1}^{\infty}$  is *irrational* if the set  $\{\sum_{n \geq 1} \frac{1}{a_n c_n} \mid c_n \in \mathbb{N}\}$ , which we refer to henceforth as its *expressible set*, contains no rational numbers. In [1] it is shown that if  $\lim_{n \rightarrow \infty} a_n^{1/2^n} = \infty$  and  $a_n \in \mathbb{N}$  for all  $n \in \mathbb{N}$  then  $\sum_{n \geq 1} a_n^{-1}$  is an irrational number. From this Erdős deduces that the sequence  $\{2^{2^n}\}_{n=1}^{\infty}$  is an irrational sequence. Thus its expressible set contains no rational numbers. In [2] it is shown that if  $a_n \in \mathbb{R}^+$  for all  $n \in \mathbb{N}$  and  $\limsup_{n \rightarrow \infty} \frac{1}{n} \log_2 \log_2 a_n < 1$  then the expressible set of the sequence  $\{a_n\}_{n=1}^{\infty}$  contains an interval. It seems to be the case that in general finding the expressible set for the sequence  $\{a_n\}_{n=1}^{\infty}$  is not easy.

Ein interessantes zahlentheoretisches Problem ist die Frage nach der Rationalität des Werts einer konvergenten Reihe reeller Zahlen. An diese Fragestellung anknüpfend nennen wir mit P. Erdős eine Folge  $\{a_n\}_{n=1}^{\infty}$  reeller Zahlen irrational, falls die Menge  $E = \{\sum_{n=1}^{\infty} 1/(a_n c_n) \mid c_n \in \mathbb{N}\}$  keine rationale Zahl enthält. In der vorliegenden Arbeit beweisen die Autoren für den Fall, dass die Reihe  $\sum_{n=1}^{\infty} 1/a_n$  bedingt konvergent ist, dass die Menge  $E$  jeweils die gesamte reelle Zahlengerade ausschöpft.

In this paper we give conditions on  $\{a_n\}_{n=1}^{\infty}$  to ensure that its expressible set is equal to  $\mathbb{R}$ . We prove the following:

**Theorem 1.** *Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of nonzero real numbers such that the series  $\sum_{n=1}^{\infty} \frac{1}{a_n}$  is conditionally convergent. Then its expressible set is equal to  $\mathbb{R}$ .*

A series is conditionally convergent if it is convergent but the series of the absolute values of its terms is not. Theorem 1 is an immediate consequence of the following more general theorem.

**Theorem 2.** *Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of nonzero real numbers such that the series  $\sum_{n=1}^{\infty} \frac{1}{a_n}$  is conditionally convergent. Then for every pair  $\alpha, \beta$  of real numbers with  $\alpha \leq \beta$  there exists a sequence  $\{c_n\}_{n=1}^{\infty}$  of positive integers such that*

$$\alpha = \liminf_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{a_n c_n} \quad \text{and} \quad \beta = \limsup_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{a_n c_n}. \quad (1)$$

For the proof of Theorem 2 we need the following two lemmas.

**Lemma 1.** *Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of nonzero real numbers such that the series  $\sum_{n=1}^{\infty} \frac{1}{a_n}$  is conditionally convergent. Then for every real number  $A \geq 0$  and every integer  $N \geq 0$  there exist a number  $K \in \mathbb{N}$  and numbers  $c_{N+1}, \dots, c_{N+K} \in \mathbb{N}$  such that*

$$\sum_{n=N+1}^{N+K} \frac{1}{a_n c_n} \in \left( A, A + \frac{1}{a_{N+K}} \right].$$

*Proof.* Define  $\mathcal{P} = \{n \mid a_n > 0\}$  and  $\mathcal{N} = \{n \mid a_n < 0\}$ . The series  $\sum_{n=1}^{\infty} \frac{1}{a_n}$  is conditionally convergent, hence

$$\sum_{\substack{n=N+1 \\ n \in \mathcal{P}}}^{\infty} \frac{1}{a_n} = \infty.$$

This implies that there exists a positive integer  $K$  such that

$$\sum_{\substack{n=N+1 \\ n \in \mathcal{P}}}^{N+K-1} \frac{1}{a_n} \leq A \quad \text{and} \quad \sum_{\substack{n=N+1 \\ n \in \mathcal{P}}}^{N+K} \frac{1}{a_n} > A.$$

The fact that

$$0 < \sum_{\substack{n=N+1 \\ n \in \mathcal{P}}}^{N+K} \frac{1}{a_n} - \sum_{\substack{n=N+1 \\ n \in \mathcal{P}}}^{N+K-1} \frac{1}{a_n} = \frac{1}{a_{N+K}}$$

immediately gives

$$s = \sum_{\substack{n=N+1 \\ n \in \mathcal{P}}}^{N+K} \frac{1}{a_n} \in \left( A, A + \frac{1}{a_{N+K}} \right].$$

Now consider two cases:

- (1) Assume that  $\mathcal{N} \cap \{N+1, \dots, N+K\} = \emptyset$ . In this case put  $c_n = 1$  for every  $n = N+1, \dots, N+K$  and the result follows.
- (2) Now suppose that

$$r = \sum_{\substack{n=N+1 \\ n \in \mathcal{N}}}^{N+K} \frac{1}{a_n} < 0.$$

Put  $C = \lceil \frac{r}{A-s} \rceil + 1$ . Then

$$0 > \sum_{\substack{n=N+1 \\ n \in \mathcal{N}}}^{N+K} \frac{1}{Ca_n} = \frac{1}{C} \sum_{\substack{n=N+1 \\ n \in \mathcal{N}}}^{N+K} \frac{1}{a_n} > \frac{A-s}{r} \cdot r = A-s.$$

Hence the result follows by taking  $c_n = 1$  for  $n \in \{N+1, \dots, N+K\} \cap \mathcal{P}$  and  $c_n = C$  for  $n \in \{N+1, \dots, N+K\} \cap \mathcal{N}$ .  $\square$

**Lemma 2.** Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of nonzero real numbers such that the series  $\sum_{n=1}^{\infty} \frac{1}{a_n}$  is conditionally convergent. Then for every real number  $A \leq 0$  and every integer  $N \geq 0$  there exist a number  $K \in \mathbb{N}$  and numbers  $c_{N+1}, \dots, c_{N+K} \in \mathbb{N}$  such that

$$\sum_{n=N+1}^{N+K} \frac{1}{a_n c_n} \in \left[ A - \left| \frac{1}{a_{N+K}} \right|, A \right).$$

*Proof.* Using the transformation  $a_n \mapsto -a_n$  and Lemma 1 we obtain Lemma 2.  $\square$

*Proof of Theorem 2.* In the following we set

$$S_k = \sum_{n=1}^k \frac{1}{a_n c_n}.$$

If  $\beta \geq 0$  then putting  $A = \beta$  and  $N = 0$  into Lemma 1 we obtain a number  $K$  and a sequence  $\{c_n\}_{n=1}^K$  such that

$$S_K \in \left( \beta, \beta + \frac{1}{a_K} \right].$$

Then set  $N_0 = 0$  and  $N_1 = K$ .

Similarly, if  $\beta < 0$  then  $\alpha < 0$ , and putting  $A = \alpha$  and  $N = 0$  into Lemma 2 we get  $K$  and  $\{c_n\}_{n=1}^K$  with

$$S_K \in \left[ \alpha - \left| \frac{1}{a_K} \right|, \alpha \right).$$

Then set  $N_0 = K$ .

Now we will construct the sequence  $\{c_n\}_{n=1}^{\infty}$  by induction. Consider two cases:

(1) Suppose that we have constructed the sequence  $\{N_m\}_{m=0}^{2t+1}$ ,  $t \in \mathbb{N}_0$ , with

$$S_{N_{2t+1}} \in \left( \beta, \beta + \frac{1}{a_{N_{2t+1}}} \right].$$

Lemma 2 implies that there exist  $K$  and  $\{c_n\}_{n=N_{2t+1}+1}^{N_{2t+1}+K}$  such that

$$\sum_{n=N_{2t+1}+1}^{N_{2t+1}+K} \frac{1}{a_n c_n} \in \left[ \alpha - S_{N_{2t+1}} - \left| \frac{1}{a_{N_{2t+1}+K}} \right|, \alpha - S_{N_{2t+1}} \right).$$

Let  $N_{2t+2} = N_{2t+1} + K$ . Then we have

$$S_{N_{2t+2}} \in \left[ \alpha - \left| \frac{1}{a_{N_{2t+2}}} \right|, \alpha \right).$$

(2) Suppose that we have constructed the sequence  $\{N_m\}_{m=0}^{2t}$ ,  $t \in \mathbb{N}_0$ , with

$$S_{N_{2t}} \in \left[ \alpha - \left| \frac{1}{a_{N_{2t}}} \right|, \alpha \right).$$

Lemma 1 implies that there exist  $K$  and  $\{c_n\}_{n=N_{2t}+1}^{N_{2t}+K}$  such that

$$\sum_{n=N_{2t}+1}^{N_{2t}+K} \frac{1}{a_n c_n} \in \left( \beta - S_{N_{2t}}, \beta - S_{N_{2t}} + \frac{1}{a_{N_{2t}+K}} \right).$$

Let  $N_{2t+1} = N_{2t} + K$ . Then we have

$$S_{N_{2t+1}} \in \left( \beta, \beta + \frac{1}{a_{N_{2t+1}}} \right].$$

Using alternatively cases (1) and (2) we construct the whole sequence  $\{c_n\}_{n=1}^{\infty}$ . From the construction it follows that

- $\alpha - \left| \frac{1}{a_k} \right| \leq S_k \leq \beta + \left| \frac{1}{a_k} \right|$  for every  $k \geq N_1$ ,
- $S_{N_{2t}} < \alpha$  for every  $t \in \mathbb{N}$ ,
- $S_{N_{2t+1}} > \beta$  for every  $t \in \mathbb{N}_0$ .

The series  $\sum_{n=1}^{\infty} \frac{1}{a_n}$  is conditionally convergent, hence  $\frac{1}{a_n} \rightarrow 0$ . This implies that

$$\lim_{t \rightarrow \infty} S_{N_{2t}} = \alpha \quad \text{and} \quad \lim_{t \rightarrow \infty} S_{N_{2t+1}} = \beta$$

and the result follows. □

**References**

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