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1 Introduction

We consider the Fibonacci sequence $(F_n)_{n\geq 0}$ and the Lucas sequence $(L_n)_{n\geq 0}$ both of which are solutions to the linear recurrence $u_{n+2} = u_{n+1} + u_n$, with the initial conditions $F_0 = 0$, $F_1 = 1$ and, respectively, $L_0 = 2$, $L_1 = 1$.

Das Problem, alle reinen Potenzen in der Fibonacci- und der Lucas-Folge zu finden, wurde vor kurzem von drei der vier Verfasser der vorliegenden Arbeit gelöst. Hier geben die Autoren einerseits einen Überblick über den Beweis dieses Resultats, andererseits zeigen sie, dass die Zahlen 0, 1, 2, 3, 5 und 8 die einzigen Fibonacci-Zahlen F_n sind, für die F_n+1 oder F_n-1 eine reine Potenz ist. Dabei ist die Tatsache erstaunlich, dass das erste Ergebnis auf tiefen Resultaten, wie z.B. dem Satz von Wiles zur Modularität elliptischer Kurven (der beim Beweis der Fermat-Vermutung eine entscheidende Rolle spielte) oder der Bakerschen Theorie der linearen Formen in Logarithmen, beruht, währenddem sich das hier dargestellte neue Ergebnis relativ einfach gewinnen lässt.

The problem of determining all perfect powers in the Fibonacci sequence was a famous open problem for over 40 years, and has been resolved only recently [9].

Theorem 1. *The only perfect powers among the Fibonacci numbers are* $F_0 = 0$, $F_1 = F_2 = 1$, $F_6 = 8$ *and* $F_{12} = 144$. *For the Lucas numbers, the only perfect powers are* $L_1 = 1$ *and* $L_3 = 4$.

Subsequent papers studied several multiplicative generalizations such as $F_n = ay^p$ (see [8]) and $F_{n_1} \cdots F_{n_r} = y^p$ with $1 \leq r < p$ (see [7]). Here, we consider the (apparently) non-multiplicative question $F_n \pm 1 = y^p$. We prove the following result:

Theorem 2. *The only nonnegative integer solutions* (n, y, p) *of the equations*

$$F_n \pm 1 = y^p$$

with $p \geq 2$ *are*

$$F_0 + 1 = 0 + 1 = 1, \quad F_4 + 1 = 3 + 1 = 2^2, \quad F_6 + 1 = 8 + 1 = 3^2,$$
$$F_1 - 1 = F_2 - 1 = 1 - 1 = 0, \quad F_3 - 1 = 2 - 1 = 1, \quad F_5 - 1 = 5 - 1 = 2^2.$$

We note that these equations have been previously solved for $p = 2, 3$ by R. Finkelstein [14], [15], and N. Robbins [29]. In Section 2.6 of [1], J.A. Antoniadis gave an alternative resolution of $F_n - 1 = y^2$.

The traditional approach to equations involving Fibonacci numbers combines clever tricks with various elementary identities connecting Fibonacci and Lucas numbers. This is the approach we follow in proving Theorem 2. By contrast Theorem 1 was proved by combining some of the deepest tools available in number theory: namely the proof of Fermat's Last Theorem and a refined version of Baker's theory of linear forms in logarithms.

In Section 2 we discuss the modular approach (used in the proof of Fermat's Last Theorem). We also try to give the reader a feel for the modular approach through some elementary computations connected with the proof of Theorem 1. In Section 3 we sketch the main steps in the proof of Theorem 1. In Section 4 we give a brief historical survey of previous results on perfect powers in the Fibonacci sequence. Sections 5 and 6 build up to the proof of Theorem 2, which is completed in Section 7. In the final section we briefly mention a related open problem.

2 The modular approach and Fibonacci powers

In this section we would like to make a few remarks on the modular approach used in the proof of Fermat's Last Theorem. We also give the reader a feel for how the modular approach works by carrying out some very explicit and elementary calculations connected with the Fibonacci perfect powers problem. It is appropriate to point out that equations $F_n = y^p$ and $L_n = y^p$ have previously been solved for small values of the exponent p by various authors; we present a brief survey of known results in Section 4.

Wiles' proof of Fermat's Last Theorem [35], [34] is certainly the most spectacular recent achievement in the field of Diophantine equations. Although the proof is very deep, the logical structure of the proof is easy to understand. There are three main steps:

(i) Associate to a non-trivial solution of $x^p + y^p = z^p$ what is now known as a Frey elliptic curve [1]:
$$E_{x,y,z} \,:\, Y^2 = X(X + x^p)(X - y^p).$$

(ii) Ribet's Level-Lowering Theorem [28] and the Modularity Theorem [2] together imply that $E_{x,y,z}$ is associated [3] to a cuspidal newform of level 2.

(iii) There are no newforms at level 2, hence we have a contradiction.

We may attempt to apply the same strategy to other Diophantine equations. For example, sensible Frey curves are available for Diophantine equations of the form

$$ax^p + by^p = cz^p, \qquad ax^p + by^p = cz^2, \qquad ax^p + by^p = cz^3, \ \ldots \quad (p \text{ prime}).$$

If a 'sensible' Frey curve can be constructed, then we may apply step (ii) and deduce that the Frey curve is associated to a newform of a certain level N, which depends on the Diophantine equation we started with. However, whilst there are no newforms at level 2 nor at a handful of other small levels, there are newforms at all levels $N > 60$. Thus step (iii) fails in general. Several alternative strategies do apply in special cases (see for example [3], [13], [16]), though there does not seem to be a general strategy that is guaranteed to succeed.

A fact that had been underexploited is that the modular approach (when applicable) yields an infinite number of congruence conditions for the solutions of the Diophantine equation in question. Namely, for a fixed prime exponent p (which is not too small), if we choose a good prime l (all primes are good except for finitely many) then we obtain congruence conditions on x, y, z modulo l. For an explicit example of how the modular approach furnishes congruence conditions on the solutions, see below. For the above equations it is difficult to exploit this information successfully since we neither know a bound for the exponent p, nor for the variables x, y, z. This suggests that the modular approach should be applied to exponential Diophantine equations; for example, equations of the form

$$ax^p + by^p = c, \qquad ax^2 + b = cy^p, \ \ldots \quad (p \text{ prime}).$$

For such equations, Baker's theory of linear forms in logarithms (see the book of Shorey and Tijdeman [32]) gives bounds for both the exponent p and the variables x, y. This approach (through what are known as linear forms in logarithms and Thue equations) has undergone substantial refinements, though it still often yields bounds that can only be described as 'astronomical'.

[1] Non-trivial means $xyz \neq 0$. In step (i), we may suppose that x, y, z are coprime integers and p is a prime, and for technical reasons that will not concern us, we need to suppose $p > 5$, reorder the variables x, y, z and change signs so that $x \equiv -1 \pmod 4$ and $2 \mid y$.

[2] The Modularity Theorem states that all elliptic curves are modular. Wiles proved this for semi-stable elliptic curves, which was enough for the proof of Fermat's Last Theorem. Since then the proof of the Modularity Theorem has been completed in a series of papers the last of which is [5].

[3] We do not explain here what newforms are, nor the precise relationship furnished by Ribet's Theorem between Frey curves and associated newforms. We do however, later on, give an example were we explain this relationship in terms of down-to-earth congruences.

The proof of Theorem 1 marked the first time the modular approach has been combined with Baker's theory. We shortly sketch the main steps of the proof of Theorem 1 for Fibonacci numbers. Before that we illustrate the modular approach in this case by providing a few details. We are concerned with the equation $F_n = y^p$ with p prime. For technical reasons we restrict to the case $p \geq 7$. The Frey curve needed depends on the class of n modulo 6, and we restrict our discussion to $n \equiv 1 \pmod 6$. We associate to the solution (n, y, p) the Frey elliptic curve

$$E_n \;:\quad Y^2 = X^3 + L_n X^2 - X.$$

Ribet's Level-Lowering Theorem tells us that this is associated to a cuspidal newform of level 20. The only such newform itself corresponds to the elliptic curve

$$E \;:\quad Y^2 = X^3 + X^2 - X.$$

We did not explain the precise relationship between Frey curves and the newforms associated to these by Ribet's Level-Lowering Theorem. In the present context, it is easy to state the relationship in terms of very simple congruences. Let $l \neq 2, 5$ (we are excluding 2 and 5 as these are 'bad' primes in the present context). Let $N(l)$ denote the number of solutions (X, Y) to the equation E modulo l; we can write this as

$$N(l) = \#\{(X, Y) \;:\; 0 \leq X, Y \leq l-1 \text{ and } Y^2 \equiv X^3 + X^2 - X \pmod{l}\}.$$

We let $N_n(l)$ denote the corresponding quantity for E_n:

$$N_n(l) = \#\{(X, Y) \;:\; 0 \leq X, Y \leq l-1 \text{ and } Y^2 \equiv X^3 + L_n X^2 - X \pmod{l}\}.$$

The relationship between E_n and E can be expressed as follows:

(I) if $l \nmid y$ then $N_n(l) \equiv N(l) \pmod p$, and

(II) if $l \mid y$ then $N(l) \equiv -1$ or $2l + 1 \pmod p$.

To get a feel for these congruences and the information they give let us take $l = 3$. By counting we see that $N(3) = 5$. If $3 \mid y$ then (II) tells us that $5 \equiv -1$ or $7 \pmod p$; in other words $p \mid 6$ or $p \mid 2$. Both are impossible as $p \geq 7$. Hence $3 \nmid y$. By (I) we deduce that $N_n(3) \equiv 5 \pmod p$. Looking closely at the definition of N_n we see that $N_n(3)$ depends only on the congruence class of the Lucas number L_n modulo 3. A little counting tells us that:

- $L_n \equiv 0 \pmod 3$ implies $N_n(3) = 3$ and so $3 \equiv 5 \pmod p$, that is $p \mid 2$ which is impossible;
- $L_n \equiv 2 \pmod 3$ implies $N_n(3) = 1$ and so $1 \equiv 5 \pmod p$, that is $p \mid 4$ which again is impossible;
- $L_n \equiv 1 \pmod 3$ implies $N_n(3) = 5$ and so $5 \equiv 5 \pmod p$; this last case is true regardless of the value of p.

We deduce that $L_n \equiv 1 \pmod 3$. What does this tell us about n? The reader is asked to compute modulo 3 the first (say) 30 terms of the Lucas sequence L_n starting with $L_0 = 2$. Once that is done, a little reflection will convince the reader that $L_n \equiv 1 \pmod 3$ precisely when $n \equiv 1, 3, 4 \pmod 8$. However, we started out by assuming that $n \equiv 1 \pmod 6$. Thus we are now able to deduce that, if $n \equiv 1 \pmod 6$, then $n \equiv 1$ or $19 \pmod{24}$. We would in fact like to show that if $n \equiv 1 \pmod 6$ then $n = 1$. Notice that we have shown in this case that if $n > 1$ then n is at least 19. An important step in our proof of Theorem 1 is to show that if $n > 1$ then $n \geq 10^{9000}$. The following elementary exercise will give you a feel for how this is done.

Exercise. We continue with the assumptions that $p \geq 7$ and $n \equiv 1 \pmod 6$.

(a) Show that $N(7) = 5$.

(b) Make a table of values for $N_n(7)$ and deduce that $L_n \equiv 1$ or $3 \pmod 7$.

(c) Show $L_n \equiv 1$ or $3 \pmod 7$ implies $n \equiv 1, 2, 7, 11, 13, 14 \pmod{16}$.

(d) But we know from the above that $n \equiv 1$ or $19 \pmod{24}$. Deduce from this and (c) that $n \equiv 1$ or $43 \pmod{48}$.

Note from part (d) of the exercise that if $n > 1$ then $n \geq 43$. Before all we could say was $n \geq 19$. Thus by considering one value of l we have been able to increase our lower bound for n by a factor of $43/19 \approx 2.26$.

3 Scheme of the proof of Theorem 1

The main steps in the proof of Theorem 1 for Fibonacci numbers are as follows (the case of Lucas numbers is similar, and in fact simpler):

(i) We associate Frey curves to putative solutions of the equation $F_n = y^p$ with even index n and apply the modular approach. This, together with some elementary arguments is used to reduce to the case where the index n satisfies $n \equiv \pm 1 \pmod 6$.

(ii) We then show that we may suppose that the index n in the equations $F_n = y^p$ is prime: this is essentially a result proved first by Pethő [25] and Robbins [30] (independently).

(iii) Using Binet's formulæ – see (1) below – one sees at once that the equation $F_n = y^p$ implies that the linear form

$$\Lambda = n \log \alpha - \log \sqrt{5} - p \log y$$

is very small (here and below we write $\alpha = (1 + \sqrt{5})/2$). Then a lower bound for linear forms in logarithms gives an upper bound on the exponent p. Applying a powerful improvement to known bounds for linear forms in three logarithms we get that $p < 2 \times 10^8$.

(iv) Knowing that $p < 2 \times 10^8$ in the Fibonacci case, we apply the modular approach again under the assumption that the index n is odd. We are able to show, using the congruences given by the modular approach, that $n \equiv \pm 1 \pmod p$.

(v) As seen in step (iii), the equation $F_n = y^p$ yields a linear form in three logarithms. However we know that $n \equiv \pm 1 \pmod{p}$. In this case the linear form in three logarithms may be easily rewritten as a linear form in two logarithms. For example, if $n = kp + 1$, then we can rewrite Λ as

$$\Lambda = p \log(\alpha^k / y) + \log(\alpha/\sqrt{5}).$$

The bounds available for linear forms in two logarithms are substantially better than those available for linear forms in three logarithms. Applying [17] we deduce that $p \le 733$. A serious improvement!

(vi) We reduce the equations $F_n = y^p$ to Thue equations; these are equations of the form $G(u, v) = 1$ where G is a binary form of degree p. We do not solve these Thue equations completely, but we compute explicit upper bounds for their solutions using classical methods (see for example [6]). This provides us with upper bounds for n in terms of p. To be precise, we prove that $n < 10^{9000}$, which is a rather large bound for an index.

(vii) We show how the congruences given by the modular approach can be used, with the aid of a computer program, to produce extremely stringent congruence conditions on n. For $p \le 733$ in the Fibonacci case, the congruences obtained are so strong that, when combined with the upper bounds for n in terms of p obtained in (iv), they give a complete resolution for $F_n = y^p$.

Let us make some brief comments.

The condition $n \equiv \pm 1 \pmod{p}$ obtained after step (iv) cannot be strengthened. Indeed, we may define F_n and L_n for negative n by the recursion formulæ $F_{n+2} = F_{n+1} + F_n$ and $L_{n+2} = L_{n+1} + L_n$. We then observe that $F_{-1} = 1$ and $L_{-1} = -1$. Consequently, F_{-1}, F_1, L_{-1} and L_1 are p-th powers for any odd prime p. Thus equations $F_n = y^p$ and $L_n = y^p$ do have solutions with $n \equiv \pm 1 \pmod{p}$.

The computations in the paper were performed using the computer packages `PARI/GP` [2] and `MAGMA` [4]. The total running time for the various computational parts of the proof of Theorem 1 was about a week.

4 A brief survey of previous results

In this section we give a very brief survey of results known to us on the problem of perfect powers in the Fibonacci and Lucas sequences, though we make no claim that our survey is exhaustive.

Before stating specific results on Fibonacci and Lucas numbers, we note that Pethő [24] and, independently, Shorey and Stewart [31] proved that there are only finitely many perfect powers in any non-trivial binary recurrence sequence. Their proofs, based on Baker's theory of linear forms in logarithms, are effective but yield huge bounds. We now turn to specific results on the Fibonacci and Lucas sequences.

- The only perfect squares in the Fibonacci sequence are $F_0 = 0$, $F_1 = F_2 = 1$ and $F_{12} = 144$; this is a straightforward consequence of two papers by Ljunggren [18],

[19] (see also [21]). This has been rediscovered by Cohn [11] (see the Introduction to [20]) and Wyler [36].

- London and Finkelstein [22] showed that the only perfect cubes in the Fibonacci sequence are $F_0 = 0$, $F_1 = F_2 = 1$ or $F_6 = 8$. This was reproved by Pethő [25], using a linear form in logarithms and congruence conditions.
- For $m = 5, 7, 11, 13, 17$, the only m-th powers are $F_0 = 0$, $F_1 = F_2 = 1$. The case $m = 5$ is due to Pethő [26], using the method described in [25]. It has been reproved by McLaughlin [23] by using a linear form in logarithms together with the LLL algorithm. The other cases are solved in [23] with this method.
- If $n > 2$ and $F_n = y^p$ then $p < 5.1 \times 10^{17}$; this was proved by Pethő using a linear form in three logarithms [27]. In the same paper he also showed that if $n > 2$ and $L_n = y^p$ then $p < 13222$ using a linear form in two logarithms.
- Another result which is particularly relevant to the proof of Theorem 1 is the following: If $p \geq 3$ and $F_n = y^p$ for an integer y then either $n = 0, 1, 2, 6$ or there is a prime $q \mid n$ such that $F_q = y_1^p$, for some integer y_1. This result was established by Pethő [25] and Robbins [30] independently.
- Cohn [12] proved that $L_1 = 1$ and $L_3 = 4$ are the only squares in the Lucas sequence.
- London and Finkelstein [22] proved that $L_1 = 1$ is the only cube in the Lucas sequence.
  The proof of London and Finkelstein consists of solving some elliptic equations obtained by combining (2) below with the conditions $F_n = y^3$ and $L_n = z^3$. It is also interesting to note – as London and Finkelstein have done – that a paper of Siegel [33] shows that determining all the cubes among Fibonacci and Lucas numbers gives a new solution of the old famous problem of determining all the imaginary quadratic fields with class-number one; see also a more recent paper of Chen on this subject [10].

We end this section with very elementary results which will be useful throughout the rest of this paper. We use Binet's formulæ

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad L_n = \alpha^n + \beta^n, \tag{1}$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, that imply the following well-known relation between F_n and L_n.

Lemma 1. *For any integer n, the Fibonacci and Lucas numbers F_n and L_n satisfy the quadratic relation*

$$L_n^2 - 5F_n^2 = 4\,(-1)^n. \tag{2}$$

This quickly leads us to associate the equations $F_n = y^p$ and $L_n = y^p$ with auxiliary equations of the type $ax^2 - b = cy^p$ discussed above as examples for which the modular method may be applied.

5 Factorization

As noted previously, the Fibonacci and Lucas sequences can be extrapolated backwards using $F_n = F_{n+2} - F_{n+1}$ and $L_n = L_{n+2} - L_{n+1}$. Thus, for example, $F_{-1} = 1$, $F_{-2} = -1$, and so on. Binet's formulæ (1) remain valid for Fibonacci and Lucas numbers with negative indices, and they allow us to show easily that

$$F_a\, L_b = F_{a+b} + (-1)^b F_{a-b} \tag{3}$$

for any two integers a, b. We use this to turn the equation $F_n + 1 = y^p$ into a multiplicative instead of an additive problem. Here we are helped by the fact that $F_{-1} = F_1 = F_2 = 1$ and $F_{-2} = -1$. A little experimentation gives a different factorization for $F_n + 1$ depending on the class of n modulo 4:

$$F_{4k} + 1 = F_{2k-1}L_{2k+1}, \qquad F_{4k+1} + 1 = F_{2k+1}L_{2k}, \tag{4}$$

$$F_{4k+2} + 1 = F_{2k+2}L_{2k}, \qquad F_{4k+3} + 1 = F_{2k+1}L_{2k+2}. \tag{5}$$

Thus we are led to consider four equations of the form $F_a L_b = y^p$. If the Fibonacci and Lucas numbers in question are coprime we instantly deduce that both are perfect powers and conclude using Theorem 1. This is not true in all the cases we require; the next section provides the necessary information on the greatest common divisors of these Fibonacci and Lucas numbers.

6 Common factors of Fibonacci and Lucas numbers

The following are well-known facts whose proofs we sketch for the convenience of the reader.

Lemma 2. *The following properties hold for all nonnegative integers n:*

1) $\gcd(F_{n+1}, F_n) = 1$;
2) $\gcd(F_{n+2}, F_n) = 1$;
3) 3 *divides* F_n *if and only if* 4 *divides* n;
4) $\gcd(F_{n+2}, 3F_n)$ *is* 1 *if* 4 *does not divide* $n+2$, *and is* 3 *otherwise;*
5) $\gcd(3F_{n+2}, F_n)$ *is* 1 *if* 4 *does not divide* n, *and is* 3 *otherwise;*
6) 2 *divides* F_n *if and only if* 3 *divides* n.

Proof. (Sketch)
1) The Euclidean algorithm with input F_{n+1} and F_n gives the sequence F_{n+1}, F_n, F_{n-1}, $\ldots$, $F_1 = 1$; hence, the result. Moreover, this is the "slowest" example for the Euclidean algorithm. This is Lamé's Theorem, proved around 1830.

2) Follows from 1) and the relation $F_{n+2} = F_{n+1} + F_n$.

3) Computing the sequence $(F_n)_{n\geq 0}$ modulo 3 one notes that the period is 8.

4) By 2), the greatest common divisor of the two numbers is 1 when 3 does not divide F_{n+2} and is 3 otherwise. The desired conclusion follows from 3).

5) Similar to 4).

6) Exercise. □

Lemma 3. *For all nonnegative integers n we have:*

1) $\gcd(F_n, L_n)$ *is* 1 *if* 3 *does not divide n, and is* 2 *otherwise;*
2) $\gcd(F_{n+1}, L_n) = \gcd(L_{n+1}, F_n) = 1$*;*
3) $\gcd(F_{n+2}, L_n)$ *is* 1 *if* 4 *does not divide* $n+2$*, and is* 3 *otherwise;*
4) $\gcd(F_{n-2}, L_n)$ *is* 1 *if* 4 *does not divide* $n-2$*, and is* 3 *otherwise.*

Proof. The proof follows easily from Lemma 2 and the relations

$$L_n = 2F_{n+1} - F_n = 2F_{n-1} + F_n = 2F_{n+2} - 3F_n = -2F_{n-2} + 3F_n,$$

which can be obtained almost directly from Binet's formulæ (1) and the defining relation $F_{n+2} = F_{n+1} + F_n$. □

7 Proof of Theorem 2

We now return to equation $F_n + 1 = y^p$. We know from (4) and (5) that $F_n + 1 = F_a L_b$ where the pair of integers a, b depends on the class of n modulo 4. By Lemma 3, the greatest common divisor of the two factors in the above products is always 1 except when $n \equiv 6 \pmod 8$, in which case it is equal to 3. Since we already know the solutions of $F_n = y^p$ and $L_n = y^p$ for $p \geq 2$ (Theorem 1), we only have to consider the equation $F_a = 3^k y^p$. The result for $F_n + 1 = y^p$ follows from the following proposition.

Proposition 1. *The only positive integer solutions* (n, k, p, y) *to the equation*

$$F_n = 3^k y^p \quad \text{with} \quad k > 0 \quad \text{and} \quad p \geq 2$$

are $F_4 = 3 \cdot 1$ *and* $F_{12} = 3^2 \cdot 4^2$.

Proof. By considering the Fibonacci sequence modulo 3 and 9 it is easy to see that $3 \mid F_n$ if and only if $4 \mid n$, and $9 \mid F_n$ if and only if $12 \mid n$. Suppose that $F_n = 3^k y^p$ with $k > 1$. Then 3 divides n and, by Lemma 4 below, $F_{n/3} = 3^{k-1} z_1^p$ with some positive integer z_1. So, we treat first the case when $k = 1$ and z is not a multiple of 3. Since 3 divides F_n, we get that $n = 4h$, where 3 does not divide h because 3 does not divide z. Then $F_{4h} = F_{2h} L_{2h}$, where F_{2h} and L_{2h} are coprime. Hence, $F_{2h} = t^p$ or $L_{2h} = t^p$ and Theorem 1 implies that $h = 1$. The conclusion is now immediate by noticing that F_{36} is not a solution. □

Lemma 4. *For all nonnegative integer n,*

$$F_{3n} = F_n\left(5F_n^2 + (-1)^n 3\right) = F_n Z_n.$$

Furthermore, $\gcd(F_n, Z_n) = 3$ *when* 3 *divides* F_n*. Moreover,* 9 *never divides* Z_n*.*

Proof. Exercise. □

We leave it as an exercise to the reader to discover the necessary factorizations of $F_n - 1$ using (3) and to complete the proof of Theorem 2 by solving $F_n - 1 = y^p$.

8 An open problem

We conclude by posing an open problem. Find all the solutions to the equation

$$F_n + 2 = y^p, \qquad p \geq 2.$$

For odd n it is possible to factorize $F_n + 2$ and solve this problem; but no such factorization is known for even n.

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