❙ Elemente der Mathematik

Compromise, consensus, and the iteration of means

Ulrich Krause

Ulrich Krause received his doctoral degree in mathematics from the University of Erlangen-Nürnberg in 1967 and a doctoral degree in economics from the University of Hannover in 1979. His interests include the theory of dynamical systems and commutative algebra as well as the economics of production and social welfare theory. Until 2006 he held the position of a professor of mathematics at the University of Bremen.

1 Introduction

Intuitively, making a compromise within a dispute means to take the opinions of others into account. The hidden hope thereby is to approach a consensus. By what weights, however, should different people be taken into account? Does one really approach a consensus by making compromises – or could one run into a dissent or maybe separate forever? Giving weights to others could be mathematically modelled by a weighted mean – but which one? There are a large variety of means, including the arithmetic mean, the geometric mean, the harmonic mean, various power means etc. Could it be that the question of approaching a consensus does depend on the kind of mean applied?

Es wird ein einfaches Modell vorgestellt und analysiert für die kollektive Dynamik einer Gruppe von Akteuren, die zu echten Kompromissen bereit sind. Interessante Beispiele sind etwa die Meinungsbildung unter mehreren Individuen oder die Koordination zwischen autonomen Robotern. Dabei wird angenommen, dass der Vorschlag $x^i(t+1)$ des Akteurs i eine individuell festgelegte konvexe Kombination der auf dem Tisch liegenden Gebote $x^1(t), \ldots, x^n(t) \in \mathbb{R}^d$ ist. Für diesen Kompromiss-Algorithmus konvergieren bei $t \to \infty$ die Vorschläge $x^i(t)$, $1 \le i \le n$, gegen einen gemeinsamen Wert, den Konsens, der auch noch von den Startwerten $x^1(0), \ldots, x^n(0)$ abhängen kann. Einen linearen Spezialfall dieses Modells stellt das ergodische Verhalten primitiver Markovketten dar. Ein nichtlinearer Spezialfall ist das arithmetisch-geometrische Mittel von Gauß aus dem Jahr 1799. Dieses lässt sich deuten als eine Meinungsbildung unter zwei Akteuren, wobei der eine jeweils das arithmetische, der andere das geometrische Mittel der letzten Gebote als Kompromissvorschlag wählt.

For simplicity, imagine two persons with initial opinions represented by two different positive numbers a and b, respectively. These numbers may be thought of as two different assessments of a certain magnitude by the two persons. Suppose, person 1 compromises for certain reasons and changes her opinion to the arithmetic mean $a' = \frac{a+b}{2}$, whereas person 2 compromises by changing b to the geometric mean $b' = \sqrt{ab}$. In general, the numbers a' and b' will not yet be equal and, hence, opinions will be changed anew. By iteration one obtains two sequences of positive numbers a_k and b_k, respectively. As early as 1799 it was observed by Gauss that the two sequences converge to a joint limit c, the arithmetic-geometric mean, which is given by a complete elliptic integral [1, p. 5]. This might be interpreted by saying that the two persons compromising as above will tend to a consensus c, which depends on the initial opinions a and b via the integral.

Since the above algorithm converges very fast, it is used in practical computations of elliptic integrals. Following Gauss (and Legendre), many variations and extensions of the algorithm have been investigated (cf. [1], [2]). Thereby, the convergence is the easier issue compared to the determination of the value of the consensus. The latter task can be very difficult and is still open even in seemingly simple cases. In the literature on Gauss' arithmetic-geometric mean, extensions to n numbers instead of just two numbers a and b have been considered as well as other means, like (weighted) power means (see Section 3). For a rather general mean which is an abstraction of the known concrete means, the so called *abstract mean*, convergence was proved in [1, p. 244]. As the authors put it: "There is a great literature on particular means and very little on means in general." [1, p. 235].

All the means considered so far, including the abstract mean, are means of finitely many real numbers. These means may model the making of a compromise among finitely many persons expressing one dimensional opinions. In this article we take up more generally the issue of forming means of points in higher dimensional space. Considering the process to compromise, this means that we admit the persons to express more refined opinions, represented by bundles of real numbers. Forming a mean now amounts to forming a convex combination in higher dimensional space. This general framework does not only cover the classical issue of mean iteration but also the more recent models of opinion formation and multiagent communication, developed for organisms in biology as well as for decision-takers in economics and robots in engineering. Following a common usage we will, therefore, use the term "agent" instead of "person". (For the models mentioned and their history see [3], [4], [5] and the references given therein.)

Section 2 presents a model of compromising behavior for finitely many agents. There is no need to go into the motives of the agents which are often opaque and diverse, including strategic motives. What will matter are the factual compromises taken by the agents and the dynamics of interaction. It is shown that a simple and intuitively compelling condition guarantees convergence of the dynamics to a consensus. This result extends that on abstract means in [1] and demonstrates, how consensus is approached in spite of possible conflicts between different dimensions. The proof of the result is elementary and needs no elaborated tools.

Section 3 illustrates the general convergence result by examples. Surprisingly, the famous ergodic theorem for primitive Markov chains results easily by specializing to linear compromise maps. Convergence results, known from the literature, can be obtained in a

sharpened version. For a nonlinear model of opinion dynamics a criterion for consensus can be derived.

Section 4 contains concluding remarks and discusses a simple example, which is not covered by criteria known hitherto.

2   Convergence of the compromise algorithm to consensus

Consider a number n of agents involved in a process of compromising when taking actions. Let S denote the set of all agents' possible actions. For agent $i \in \{1, \ldots, n\}$ denote by $f_i(x^1, x^2, \ldots, x^n)$ the action she takes in the next period by making a compromise based on the actions $x^1, x^2, \ldots, x^n$ of all agents (including herself) in the previous period.

The idea now is to model such a compromise as a convex combination of $x^1, x^2, \ldots, x^n$. For this we take the action set S to be a non-empty *convex subset* of d-dimensional real space $\mathbb{R}^d$, that is with any two points u and v the set S contains also the convex combination $\alpha u + (1 - \alpha)v$ for each $\alpha \in [0, 1]$. It follows that for $x^i \in S$, $1 \leq i \leq n$, the *convex hull* $\mathrm{conv}\{x^1, x^2, \ldots, x^n\}$, consisting of all convex combinations $\alpha_1 x^1 + \alpha_2 x^2 + \ldots + \alpha_n x^n$ with $0 \leq \alpha_i$ for all i and $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$ is contained in S, too. Collecting the mappings f_i of the agents in $f = (f_1, \ldots, f_n)$, the mapping $f \colon S^n \to S^n$ is called a *compromise map* on S for n agents if

$$\mathrm{conv}\{f_1(x), \ldots, f_n(x)\} \subseteq \mathrm{conv}\{x^1, \ldots, x^n\} \tag{1}$$

holds for all $x = (x^1, \ldots, x^n) \in S^n$.

We call a compromise map *proper* if for $x^1, \ldots, x^n$, not all equal, the inclusion in (1) holds properly. That is "$\subseteq$" holds but not "$=$".

Since in one dimension convex sets coincide with intervals (including the empty set, a point, or infinite intervals), in the special case of $d = 1$ condition (1) amounts to

$$\min_{1 \leq j \leq n} x^j \leq f_i(x) \leq \max_{1 \leq j \leq n} x^j \tag{2}$$

for all $1 \leq i \leq n$, all $x = (x^1, \ldots, x^n) \in S^n \subseteq \mathbb{R}^n$.

A continuous map f_i which satisfies (2) is called an *abstract mean*. An abstract mean is called *strict* [1, p. 230] if in (2) both inequalities are strict for unequal $x^1, \ldots, x^n$.

The reader will easily confirm that examples of strict abstract means are given by arithmetic mean and geometric mean as well as by any other of the common concrete means.

Another interesting example of a compromise map appears in *opinion dynamics* (see [3], [4]). For a profile $x = (x^1, \ldots, x^n)$ of the agents' opinions x^j (maybe multidimensional) select a subset $I(i, x)$ of all agents which contains i, as a neighborhood of agent i at profile x. Define $f_i(x)$ as the arithmetic mean of the opinions of neighbors of agent i, that is

$$f_i(x) = \frac{1}{\sharp I(i, x)} \sum_{j \in I(i,x)} x^j,$$

where "$\sharp$" denotes the number of elements of a finite set. Of particular interest is a neighborhood $I(i, x)$, which consists of all agents j with $\|x^i - x^j\| \leq \epsilon$ for some confidence

level ϵ and some norm $\| \cdot \|$ on $\mathbb{R}^d$. The compromise map $f = (f_1, \ldots, f_n)$ is a nonlinear map, the iterates of which are analytically very difficult to handle (see [3]). Instead of using an arithmetic mean one could use some other mean to average the opinions in a neighborhood (see [4]).

Obviously, abstract means $f_1, f_2, \ldots, f_n$ define a special compromise map f, which is proper if all f_i are strict. It should be noted, however, that f can be proper though none of the f_i is strict (see Section 4 for an example).

As mentioned in the introduction, the point to consider a compromise map is the possible emergence of consensus by iterating the map. Obviously, consensus is impossible in general if equality holds in condition (1). Therefore, the assumption of a proper compromise map will be important. The following result states that a proper and continuous compromise map leads by iteration always to a consensus. The proof is elementary, in that it uses only common properties of convex and compact sets in $\mathbb{R}^d$.

Theorem. *For a proper and continuous compromise map f the compromise algorithm given by the recursion*

$$x^i(t+1) = f_i(x^1(t), \ldots, x^n(t)) \text{ for } 1 \leq i \leq n, \ t \in \{0, 1, 2, \ldots\}, \qquad (3)$$

converges always to a consensus γ. That is, for some $\gamma = \gamma(x(0))$

$$\lim_{t \to \infty} x^i(t) = \gamma \text{ for all } 1 \leq i \leq n, \ x(0) = (x^1(0), \ldots, x^n(0)) \in S^n. \qquad (4)$$

Proof. Fix $x(0) = (x^1(0), \ldots, x^n(0))$ and let $C(t) = \text{conv}\{x^1(t), \ldots, x^n(t)\}$ for $t \geq 0$. Since f is a compromise map, it follows that $C(t+1) \subseteq C(t)$. Therefore, $(C(t))_{t \geq 0}$ is a decreasing sequence of convex compact subsets of $\mathbb{R}^d$ and, hence, $C = \bigcap_{t=0}^{\infty} C(t)$ is non-empty convex and compact.

(1)   Since $x(t) \in C(t)^n \subseteq C(0)^n$ for all $t \geq 0$, there exists a convergent subsequence $y(s) = x(t_s)$, $s \geq 0$, with $\lim_{s \to \infty} y(s) = c = (c^1, \ldots, c^n) \in C(0)^n \subseteq S^n$. We shall show that

$$C = \text{conv}\{c^1, \ldots, c^n\}. \qquad (5)$$

Obviously, $c^i = \lim_{s \to \infty} x^i(t_s) \in C(t)$ for every $t \geq 0$ and, hence, $c^i \in C$. Since C is convex, one has that $\text{conv}\{c^1, \ldots, c^n\} \subseteq C$. For the remaining conclusion, let $x \in C$ and choose $\delta > 0$. There exists s_0 such that $\|x^i(t_s) - c^i\| \leq \delta$ for all $s \geq s_0$ and all $1 \leq i \leq n$ (pick $\| \cdot \|$ to be any norm on $\mathbb{R}^d$). From $x \in C \subseteq C(t_{s_0})$ we have $x = \sum_{i=1}^{n} \alpha_i x^i(t_{s_0})$ with $0 \leq \alpha_i$ and $\sum_{i=1}^{n} \alpha_i = 1$. This implies

$$\left\| x - \sum_{i=1}^{n} \alpha_i c^i \right\| = \left\| \sum_{i=1}^{n} \alpha_i [x^i(t_{s_0}) - c^i] \right\| \leq \sum_{i=1}^{n} \alpha_i \delta = \delta. \qquad (6)$$

Since $\delta > 0$ is arbitrarily chosen and $\text{conv}\{c^1, \ldots, c^n\}$ is closed, we arrive at $x \in \text{conv}\{c^1, \ldots, c^n\}$. This proves (5).

(2)   Now we show that $c^1 = c^2 = \ldots = c^n$. From (3) we have

$$x^i(t_s + 1) = f_i(x^1(t_s), \ldots, x^n(t_s)) \text{ for all } s \geq 0.$$

Since $\lim_{s\to\infty} x^j(t_s) = c^j$ and f_i is continuous, we obtain that $\lim_{s\to\infty} x^i(t_s + 1) = f_i(c^1, \ldots, c^n)$. Consider now a point $x \in \text{conv}\{c^1, \ldots, c^n\}$. By step (1), $x \in C \subseteq C(t_s + 1)$ for all s and, hence,

$$x = \sum_{i=1}^{n} \alpha_i(s) x^i(t_s + 1) \quad \text{for all} \quad s,$$

where $0 \le \alpha_i(s)$ and $\sum_{i=1}^{n} \alpha_i(s) = 1$. The sequence of $(\alpha_1(s), \ldots, \alpha_n(s))$ for $s \ge 0$ is contained in the compact set $\left\{ p \in \mathbb{R}^n \mid 0 \le p_i, \ \sum_{i=1}^{n} p_i = 1 \right\}$, and, hence, there exists a sequence $(s_k)_{k\ge1}$ such that $\lim_{k\to\infty} \alpha_i(s_k) = \alpha_i^* \ge 0$ and $\sum_{i=1}^{n} \alpha_i^* = 1$.

Putting this together, we obtain

$$x = \lim_{k\to\infty} \left(\sum_{i=1}^{n} \alpha_i(s_k) x^i(t_{s_k} + 1) \right) = \sum_{i=1}^{n} \alpha_i^* f_i(c^1, \ldots, c^n).$$

Thus, $x \in \text{conv}\{f_1(c), \ldots, f_n(c)\}$ for $c = (c^1, \ldots, c^n)$. Since x was chosen arbitrarily in $\text{conv}\{c^1, \ldots, c^n\}$, this shows that we must have equality in condition (1) for $x = c$. Since f is assumed to be proper, we conclude that $c^1 = \ldots = c^n$.

(3)   By step (2) and setting $\gamma = c^i$, for $1 \le i \le n$, we know that $\lim_{s\to\infty} x^i(t_s) = \gamma$ for all i. Therefore, for $\delta > 0$ given there exists s_0 such that $\|x^i(t_s) - \gamma\| \le \delta$ for all $s \ge s_0$ and all $1 \le i \le n$. For $t \ge t_{s_0}$ it holds that $x^i(t) \in C(t) \subseteq C(t_{s_0})$ and, hence,

$$x^i(t) = \sum_{j=1}^{n} \alpha_j x^j(t_{s_0}) \quad \text{with} \quad 0 \le \alpha_j, \quad \sum_{j=1}^{n} \alpha_j = 1,$$

and where α_j may depend on i and t. Similarly, as for (6) we obtain

$$\|x^i(t) - \gamma\| = \left\| \sum_{j=1}^{n} \alpha_j [x^j(t_{s_0}) - \gamma] \right\| \le \delta.$$

Since $\delta > 0$ is arbitrary, we find that $\lim_{t\to\infty} x^i(t) = \gamma$ for all $1 \le i \le n$. Thereby, $\gamma = \gamma(x(0))$ may depend on the starting point. $\qquad\square$

3   Markov chains, Gauss iteration, and opinion dynamics

As the most simple example consider first a *linear compromise map* on $\mathbb{R}$ for n agents, that is

$$f_i(x) = \sum_{j=1}^{n} a_{ij} x^j \quad \text{with} \quad x^j \in \mathbb{R}, \quad x = (x^1, \ldots, x^n). \tag{7}$$

Thereby, the matrix A of the coefficients a_{ij} is assumed to be row-stochastic, that is the coefficients are all nonnegative with $\sum_{j=1}^{n} a_{ij} = 1$ for all $1 \le i \le n$. Replacing A by its transpose $A^\top$ the mapping $x \mapsto A^\top x$ defines a *Markov chain* (see [6]).

Consider next the example of a Gaussian mean or a *Gaussian iteration*, respectively. For the set of all strictly positive numbers S and $p_1, \ldots, p_n$ nonnegative numbers with $\sum_{i=1}^{n} p_i = 1$ define a map $f \colon S^n \to S^n$ with component maps given either by a weighted geometric mean $f_i(x) = \prod_{i=1}^{n} x_i^{p_i}$ or by a weighted power mean $f_i(x) = \left(\sum_{i=1}^{n} p_i x_i^t \right)^{\frac{1}{t}}$ for $t \in \mathbb{R}, t \neq 0$. This defines a compromise map f since the inequality (2) is satisfied.

To apply the theorem of Section 2 to these examples, we have to make sure that these compromise maps, being obviously continuous, are proper. For this consider any compromise map f for agents on a convex subset S of $\mathbb{R}$ and suppose that for each $x = (x^1, \ldots, x^n) \in S^n$ with components not all equal, the following alternative holds:

$$\min_{j} x^j < f_i(x) \leq \max_{j} x^j \quad \text{for all} \quad i \quad \text{or}$$
$$\min_{j} x^j \leq f_i(x) < \max_{j} x^j \quad \text{for all} \quad i. \tag{8}$$

It follows that $\mathrm{conv}\{f_1(x), \ldots, f_n(x)\}$ is properly contained in

$$\mathrm{conv}\{x^1, \ldots, x^n\} = \left\{ r \mid \min_{j} x^j \leq r \leq \max_{j} x^j \right\}.$$

Thus, under assumption (8) a one dimensional compromise map is proper.

To satisfy assumption (8) for the example of a linear compromise map, we assume the matrix A to be *scrambled*, that is for any two rows i and j there exists a column k, such that $a_{ik} > 0$ and $a_{jk} > 0$ (see [6]). For (8) to hold, it suffices to see that for any indices i and j the equalities $f_i(x) = \min_l x^l$ and $f_j(x) = \max_l x^l$ imply that the $x^l, 1 \leq l \leq n$, are all equal. From the two equalities we have that

$$\sum_{h=1}^{n} a_{ih} \left(x^h - \min_l x^l \right) = 0 \quad \text{and} \quad \sum_{h=1}^{n} a_{jh} \left(\max_l x^l - x^h \right) = 0.$$

By assumption, $a_{ik} > 0$ and $a_{jk} > 0$ for some k and, hence, $\min_l x^l = x^k = \max_l x^l$. Thus, all the x^l must be equal.

A similar argument applies for the example of Gauss iteration. There we must have $p_k > 0$ for at least one k. Assuming $f_i(x) = \min_l x^l$ and $f_j(x) = \max_l x^l$, it follows that $\min_l = x^k = \max_l x^l$.

As a third example consider the opinion dynamics of Section 2. Assume for the case of one dimensional opinions that at any profile x the neighborhoods $I(i, x)$ and $I(j, x)$ of any two agents have at least one agent $k = k(i, j, x)$ in common. Then, as before, from $f_i(x) = \min_l x^l$ and $f_j(x) = \max_l x^l$ it follows that

$$\sum_{h \in I(i,x)} \left(x^h - \min_l x^l \right) = 0 \quad \text{and} \quad \sum_{h \in I(j,x)} \left(\max_l x^l - x^h \right) = 0$$

and, by assumption, $\min_l x^l = x^k = \max_l x^l$. Thus, under the assumptions made in all three examples, the corresponding compromise algorithm converges to consensus. For the example of a linear compromise map we obtain, by taking as starting vectors the unit

vectors in $\mathbb{R}^n$, that $\lim_{t \to \infty} A^t = B$, where B is a matrix having all rows equal to a vector q^T. This conclusion holds also if any power of A is assumed to be scrambled. For a Markov chain given by a matrix $A^\top$ this means that $\lim_{t \to \infty} (A^\top)^t x = B^\top x = q$ for *all* initial probability distributions x, i.e., $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$. This is a fundamental theorem about Markov chains, which says that for a transition matrix having some power scrambled, the chain converges for any initial distribution to the unique stationary distribution (see [6]; there the stronger assumption is made that a power of the transition matrix is strictly positive). In this example the value of consensus is easily computed by the eigenvector q of $A^\top$.

For the Gauss iteration we obtain the convergence of various mixtures of means to a consensus, which generalizes the famous case of the arithmetic-geometric mean. (See [2] for a different proof using the stronger assumption that $p_i > 0$ for all i.) The reader will perhaps enjoy it, to verify that convergence to consensus in this example still holds for a matrix P instead of vector p as long as this matrix is scrambled. What, however, is very difficult in this example, is to find a closed formula for the consensus. Up to now, such formulas are only known for particular cases.

Concerning the example of opinion dynamics, the condition mentioned guarantees convergence of all opinions to a consensus. Though the model presented looks very simple, it seems hopeless to compute the consensus from initial conditions by some formula. Therefore, computer simulations are used extensively in analyzing opinion dynamics (see [3], [4] and references given there).

4 Concluding remarks

The compromise algorithm presented yields convergence to a consensus under the weak assumptions that the compromise map is continuous and proper. This assumption suffices to guarantee convergence for the common quantitative means as well as for Markov chains and opinion dynamics. Also, known results on Gauss iteration can be sharpened, since for an (abstract) mean condition (8) is weaker than strictness. Consider the following simple example for $n = 3$, S the set of positive real numbers and

$$f_1(x) = \frac{1}{2}x_2 + \frac{1}{2}x_3, \quad f_2(x) = \sqrt{x_1 x_3}, \quad f_3(x) = \sqrt{\frac{x_1^2 + x_2^2}{2}}.$$

Obviously, f_1, f_2, f_3 are means, but none is strict in the sense that for $x = (x_1, x_2, x_3)$ with components not all equal $f_i(x)$ lies strictly between $\min_j x_j$ and $\max_j x_j$. It is easily verified, however, that all f_i satisfy condition (8), which means that $f = (f_1, f_2, f_3)$ is a proper compromise map. Moreover, this example does not satisfy the convexity assumption in [5, p. 172], which is similar to the condition for a compromise map but does require strict convexity. (See [5, Remark 2 on p. 173]. Otherwise, the approach taken in [5] is rather general and employs, therefore, involved tools as set-valued Lyapunov theory to obtain consensus.)

Furthermore, beside the one dimensional examples discussed in Section 3, the compromise algorithm applies also to higher dimensional extensions. In particular, it would be possible to treat a compromise map, where the components are vectors from the "Gauss soup" of quantitative means across the dimensions.

Acknowledgement

References

[1] Borwein, J.M.; Borwein, P.B.: *Pi and the AGM. A Study in Analytic Number Theory and Computational Complexity.* John Wiley & Sons, New York 1987.

[2] Everett, C.J.; Metropolis, N.: A generalization of the Gauss limit for iterated means. *Adv. Math.* 7 (1971), 297–300.

[3] Hegselmann, R.; Krause, U.: Opinion dynamics and bounded confidence: models, analysis, and simulation. *J. Artificial Societies and Social Simulation* 5 (3) (2002). http://jasss.soc.surrey.ac.uk/5/3/2.html.

[4] Hegselmann, R.; Krause, U.: Opinion dynamics driven by various ways of averaging. *Comput. Economics* 25 (2005), 381–405.

[5] Moreau, L.: Stability of multiagent systems with time-dependent communication links. *IEEE Trans. Automat. Control* 50 (2005), 169–182.

[6] Seneta, E.: *Non-negative Matrices and Markov Chains.* 2nd edition. Springer, New York 1980.

Ulrich Krause
Department of Mathematics
University of Bremen, Germany
e-mail: krause@math.uni-bremen.de