Another generalisation of Napoleon's theorem

Elemente der Mathematik

Another generalisation of Napoleon's theorem

G.C. Shephard

G.C. Shephard was awarded a doctorate at Cambridge University in 1951. He held positions at the University of Birmingham and the University of East Anglia (Norwich), from which he retired as Professor Emeritus in 1987. G.C. Shephard is an expert in convexity theory, polytopes, tessellations, and the theory of patterns.

Since its discovery over 150 years ago, Napoleon's theorem has fascinated mathematicians, both professional and amateur alike. At least 150 papers have been published giving proofs, generalisations, variants and history of the theorem. Here we state and prove another generalisation.

First let us remind ourselves of the original (classical) theorem. Starting from any triangle, adjoin to each of its edges an equilateral triangle. By this we mean construct three equilateral triangles each of which has an edge in common with the original triangle. Clearly there are two ways to adjoin an equilateral triangle: either *outwardly* (in which the centres of the original triangle and the equilateral triangle lie on opposite sides of their common edge) or *inwardly* if the centres lie on the same side of the edge. Napoleon's theorem states that if all the equilateral triangles are adjoined outwardly (Fig. 1(a)), or inwardly (Fig. 1(b)), then their centres are vertices of another equilateral triangle (a *Napoleon triangle*). Most people find this result extremely surprising. It seems that the symmetries of the Napoleon triangles have "mysteriously" appeared from nowhere!

Der klassische Satz von Napoleon, der erst seit 1911 diesen Namen trägt, dürfte den meisten Lesern bekannt sein: Konstruiert man über den Seiten eines beliebigen Dreiecks jeweils gleichseitige Dreiecke und verbindet deren Mittelpunkte, so erhält man wieder ein gleichseitiges Dreieck. Es gibt eine Vielzahl von Beweisen und Verallgemeinerungen für diesen Satz. In der vorliegenden Arbeit wird folgende neue Verallgemeinerung vorgestellt: Es seien $p, q > 0$ natürliche Zahlen und n ein Teiler von $2p + q$ mit $n > p, q$. Dann bilden die beiden Mittelpunkte zweier sogenannter (n/p)-Gone über zwei Dreiecksseiten und der Mittelpunkt eines (n/q)-Gons über der dritten Dreiecksseite drei aufeinanderfolgende Ecken eines (n/p)-Gons. Der Satz von Napoleon ist durch die Spezialfälle $p = q = 1$ oder 2 gegeben.
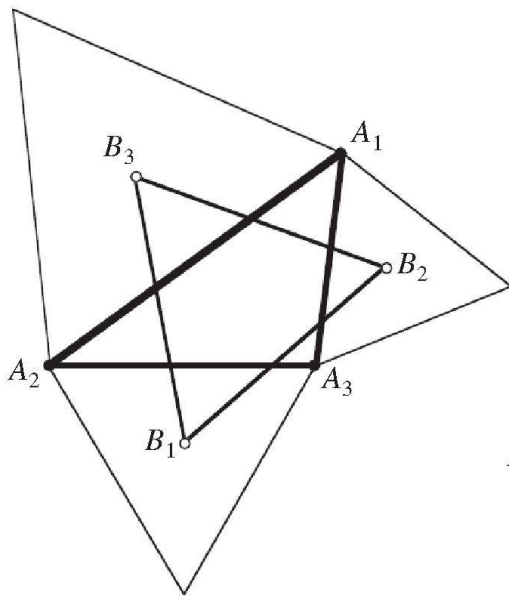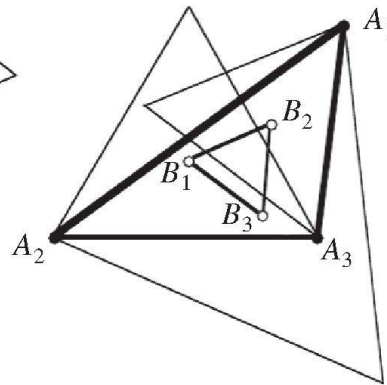
Fig. 1(a)   Fig. 1(b)

The new generalisation is as follows. Instead of adjoining equilateral triangles to the edges of an arbitrary triangle, we adjoin other regular polygons. Under certain conditions, given explicitly in the theorem, the centres of the adjoined polygons are vertices of another regular polygon. For example, if we adjoin two regular hexagons outwardly, and an equilateral triangle inwardly to the edges of an arbitrary triangle $[A_1, A_2, A_3]$ (see Fig. 3) then the centres B_1, B_2, B_3 of these three polygons are consecutive vertices of a regular hexagon.

In the following we need to extend our definition of regularity to include *regular star-polygons*. Given mutually prime integers $n \geq 3$ and $0 < p < n$ define a *regular (n/p)-gon* as a circuit (in general self-intersecting) whose n vertices are arranged equidistantly around a circle C, and whose edges subtend angles $2\pi p/n$ at the centre of C. Notice that the edges will, in general, intersect in points other that the vertices. A familiar example of a star-polygon is the *pentagram* which is a regular $(5/2)$-gon. A regular $(n/1)$-gon is the same as a regular n-gon. We note that the (internal) angle between two consecutive edges of a regular (n/p)-gon at their common vertex is $\pi(1 - 2p/n)$ and the external angle is $\pi(1 + 2p/n)$.

The centre of a regular (n/p)-gon adjoined to the edge S of a triangle T is the apex B of an isosceles triangle whose base is the edge S. Conventionally we assume that if $p/n < 1/2$, the point B and the triangle T lie on opposite sides of S, and if $p/n > 1/2$, so that the angle $2\pi p/n$ is reflex, then B and T lie on the same side of S (see Fig. 7 where the angles $\theta < \pi$ are subtended at B_1 and B_3 by the edges $[A_2, A_3]$, $[A_1, A_2]$ of the triangle, and $\phi > \pi$ is the external angle at B_2 of the triangle $[A_1, B_2, A_3]$). If $p/n = 1/2$ the polygon degenerates to a line segment which coincides with S. In this case, B is the midpoint of S.

Theorem. *Let T be any triangle $[A_1, A_2, A_3]$ and p, q be positive integers. Let n be any factor of $2p + q$ where $n > p$ and $n > q$. Let B_1, B_3 be the centres of regular (n/p)-gons*

adjoined to the edges $[A_2, A_3]$, $[A_1, A_2]$ *of* T, *and* B_2 *be the centre of a regular* (n/q)*-gon adjoined to the third edge* $[A_1, A_3]$. *Then* B_1, B_2, B_3 *are three consecutive vertices of a regular* (n/p)*-gon. In each case, if* n *and* p *or* n *and* q *have a common factor, we reduce* n/p *and* n/q *to its lowest terms.*

Notice that the conditions of the theorem imply that $(2p + q)/n = 1$ or 2. In the diagrams illustrating the following examples, the original triangle $[A_1, A_2, A_3]$ is drawn in bold lines and the regular polygon whose existence is asserted by the theorem is shown in semi-bold lines.

Examples.

1. $p = q = 1$ and $n = 3$. Here we adjoin equilateral triangles outwardly to the edges of T, and then their centres are the vertices of an equilateral triangle.

2. $p = q = 2$ and $n = 3$. Here we adjoin the equilateral triangles inwardly, and their centres are the vertices of an equilateral triangle.

Examples 1 and 2, shown in Fig. 1(a) and (b), comprise the classical Napoleon's theorem.

3. $p = 1$, $q = 2$ and $n = 4$. Two regular $(4/1)$-gons (squares) are adjoined outwardly to edges of T and the centre of the $(4/2)$-gon $(= (2/1)$-gon) is the midpoint of the third side. The centres of these three polygons are vertices of a square, see Fig. 2.
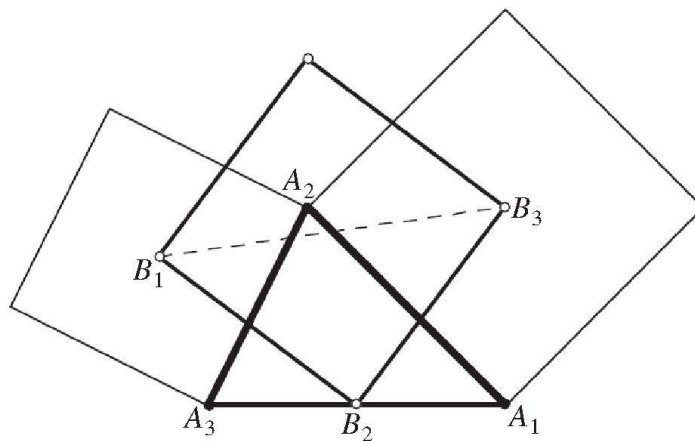

Fig. 2

4. $p = 1$, $q = 4$ and $n = 6$. The $(6/1)$-gons are regular hexagons adjoined outwardly and the $(6/4)$-gon $(= (3/2)$-gon) is an equilateral triangle adjoined inwardly. The centres of these three polygons are vertices of a regular hexagon, see Fig. 3.

5. $p = 1$, $q = 6$ and $n = 8$. Here $(8/1)$-gons (regular octagons) are adjoined outwardly and a $(8/6)$-gon $(= (4/3)$-gon) is a square adjoined inwardly. The centres of these three polygons are vertices of a regular octagon, see Fig. 4.

Examples 3, 4 and 5 are the beginning of an infinite sequence of results: for any positive integer m, the centres of regular $2m$-gons adjoined outwardly to two edges of T, and a regular m-gon adjoined inwardly to the third edge of T, are consecutive vertices of a regular $2m$-gon. This case arises by taking $p = 1$, $q = 2(m - 1)$ and $n = 2m$ in the theorem.
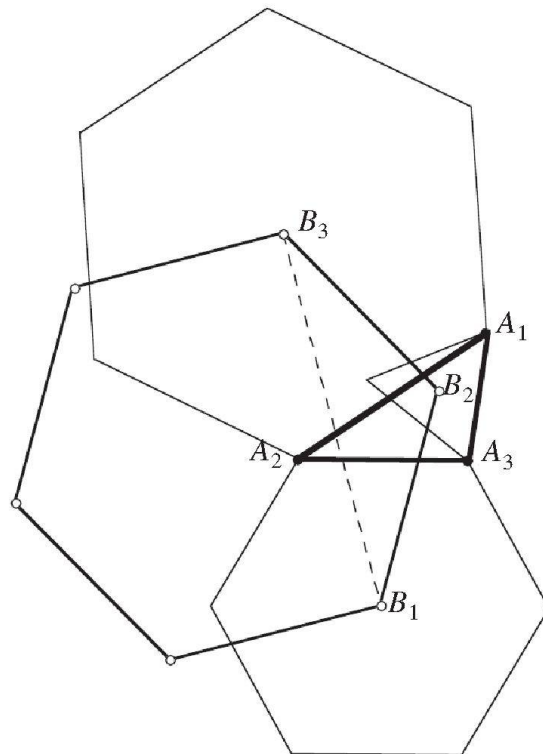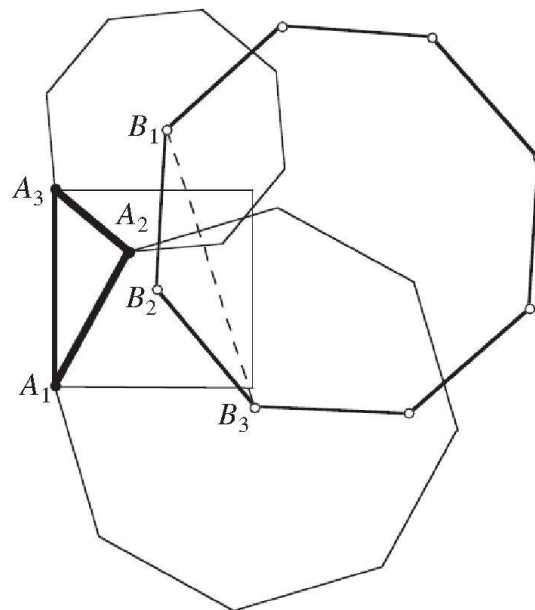
Fig. 3

Fig. 4

6. $p = 2$, $q = 1$ and $n = 5$. Two regular pentagrams ($(5/2)$-gons) are adjoined to edges of T and a regular pentagon ($(5/1)$-gon) is adjoined to the third edge. In this case all the polygons are adjoined outwardly. The centres of the adjoined polygons are three consecutive vertices of a regular pentagram, see Fig. 5.
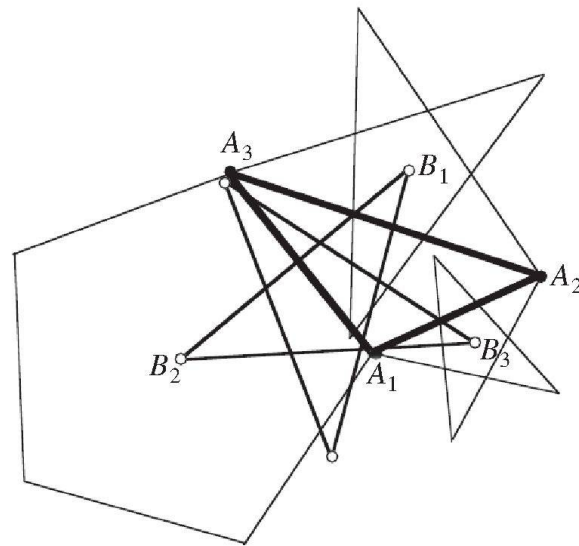
Fig. 5

7. $p = 5, q = 2$ and $n = 6$. Two regular $(6/5)$-gons (hexagons) are adjoined inwardly to the edges $[A_1, A_2]$, $[A_1, A_3]$ of T and a regular $(6/2)$-gon $(= (3/1)$-gon or equilateral triangle) is adjoined outwardly to the third edge of T. The centres of the adjoined polygons are three consecutive vertices of a regular hexagon (see Fig. 6).

Fig. 6

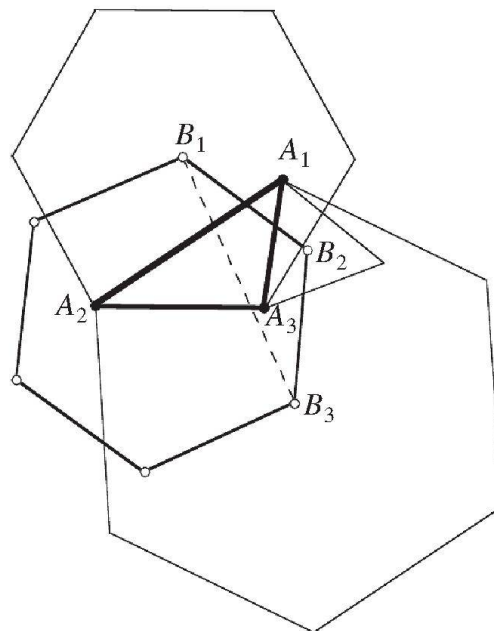*Proof of the theorem.* Let $[B_1, A_3, A_2]$, $[B_2, A_1, A_3]$, $[B_3, A_2, A_1]$ be isosceles triangles adjoined to the edges of T with apex angles θ and ϕ, as shown in Fig. 7, and suppose
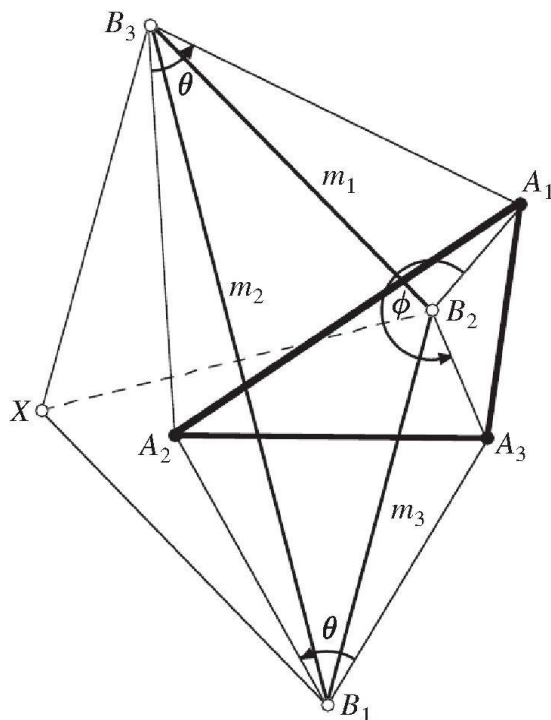
Fig. 7

$2\theta + \phi$ is a multiple of 2π. Write r_1 for rotation through angle θ about B_1, r_2 for rotation through angle ϕ about B_2, and r_3 for rotation through angle θ about B_3. Then the product $r_1r_2r_3$ is a rotation through angle $2\theta + \phi$ which is a multiple of 2π and so is a translation. But $r_3(A_2) = A_1$, $r_2(A_1) = A_3$ and $r_1(A_3) = A_2$. So A_2 is an invariant point. The only translation with an invariant point is the identity i, so $r_1r_2r_3 = i$.

Now suppose $r_1(B_2)$ is some point which we shall call X. Consider $r_3(X)$. Since $r_3^{-1} = r_1r_2$ and $r_2(B_2) = B_2$, $r_1(B_2) = X$ we deduce $r_3^{-1}(B_2) = X$ and so $r_3(X) = B_2$. The isosceles triangles $[B_3, X, B_2]$ and $[B_2, X, B_1]$ have the same angles (at B_1 and B_3) and a common base $[X, B_2]$. They are therefore congruent and $R = [X, B_1, B_2, B_3]$ is a rhomb. Its edges are equal, and, in particular $|B_3B_2| = |B_2B_1|$. If $\theta < \pi$, then the *interior* angles of R at B_1 and B_3 are each equal to θ and therefore the *interior* angle of R at B_2 is $\pi - \theta$. If $\theta > \pi$ the *exterior* angles of R at B_1 and B_3 are each equal to θ and therefore the *interior* angles at these vertices are equal to $2\pi - \theta$, and the *interior* angle at B_2 is $\theta - \pi$. Notice that the proof is valid whenever $2p + q$ is a multiple of 2π, that is if $(2p + q)/n$ is either 1 or 2. In the following we need to distinguish between these two cases.

Put $\theta = 2\pi p/n$, $\phi = 2\pi q/n$ and consider the angle between the line segments $[B_3, B_2]$ and $[B_2, B_1]$ at B_2. If $(2p + q)/n = 1$, then $\theta < \pi$ so this angle is $\pi - \theta = \pi(1 - 2p/n)$ which is the interior angle at a vertex of an (n/p)-gon. If $(2p + q)/n = 2$, so $\theta > \pi$, this angle is $\theta - \pi = \pi(2p/n - 1)$ which is also the interior angle of an (n/p)-gon. In either case, B_1, B_2, B_3 are three consecutive vertices of a regular (n/p)-gon. This proves the theorem.   □

The triangle $[B_1, B_2, B_3]$ will be called the *Napoleon triangle* for the given configuration. We observe that, since the interior angle of the triangle $[B_1, B_2, B_3]$ at B_2 is $\pi - \theta$ if $(2p + q)/n = 1$, and $\theta - \pi$ if $(2p + q)/n = 2$, in the former case, the Napoleon triangle $[B_1, B_2, B_3]$ has the *same* orientation as the triangle $T = [A_1, A_2, A_3]$ and in the latter case, it has the *opposite* orientation.

Corollary 1 *If three regular polygons satisfy the conditions of the theorem (so that their centres are vertices of a regular (n/p)-gon), then the configuration that arises by adjoining the same polygons inwardly (instead of outwardly) or outwardly (instead of inwardly) to the edges of a triangle, also has centres which are vertices of a regular (n/p)-gon.*

To prove this we apply the theorem to p', q' and n, where $p' = n - p$ and $q' = n - q$. The new configuration will be said to be *conjugate* to the original configuration. Note that if $(2p + q)/n = 1$, then for the conjugate configuration, $(2p' + q')/n = 2$. In Fig. 6 we show the configuration conjugate to that in Fig. 3, and the two configurations of Fig. 1 are conjugate to each other. Since the Napoleon triangles of a configuration and of its conjugate configuration are parts of an (n/p)-gon, we deduce they are similar, though oppositely oriented.

Given p, q as in the theorem, since the (n/q)-gon can be adjoined to any of the three edges of the triangle $[A_1, A_2, A_3]$, it follows that there are three conjugate pairs of Napoleon triangles. Hence, in all, there are six Napoleon triangles associated with the integers p, q and the given triangle.

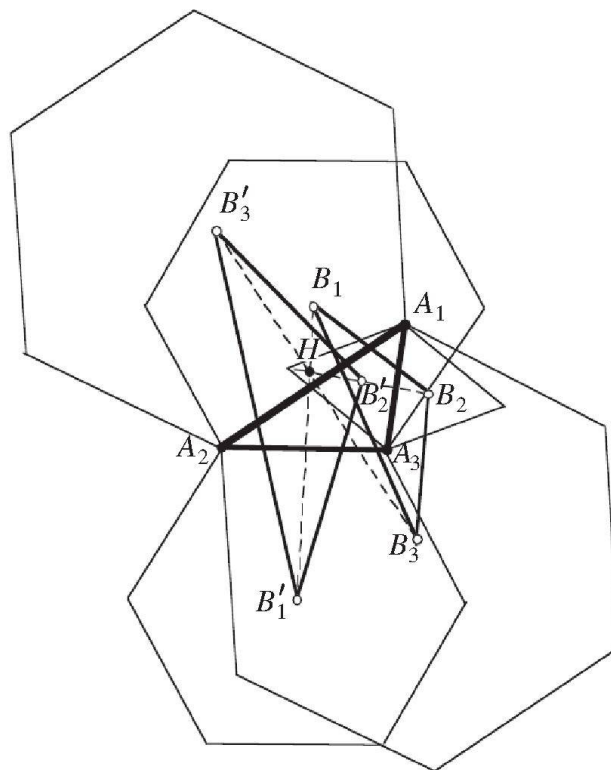
Fig. 8

Corollary 2 *These six Napoleon triangles are in perspective from the orthocentre of the original triangle* $[A_1, A_2, A_3]$.

This follows immediately from the fact that the centres of regular polygons adjoined to an edge necessarily lie on the perpendicular bisector of that edge. And these perpendicular bisectors meet at the orthocentre of $[A_1, A_2, A_3]$. In Fig. 8 we show two conjugate Napoleon triangles in perspective from H, the orthocentre of $[A_1, A_2, A_3]$. To show all six triangles would make the diagram too complicated to be intelligible.

Corollary 2 is well-known in the classical case, but note that, unlike the classical case, the original triangle $[A_1, A_2, A_3]$ is not, in general, in perspective with either of the Napoleon triangles.

Corollary 3 *If three polygons (two (n/p)-gons and an (n/q)-gon) satisfy the conditions of the theorem, then there exists a point Y through which pass the circumcircles of these three polygons.*

Again, this result is well-known for the classical Napoleon theorem. For the more general case, let m_1, m_2, m_3 be reflections in the edges $B_2 B_3$, $B_3 B_1$, $B_1 B_2$ of the triangle $[B_1, B_2, B_3]$, see Fig. 9.

Remembering that the product of reflections in two lines which intersect at angle θ is a rotation through angle 2θ, we see that, in the notation of the theorem, $r_1 = m_2 m_3$, $r_2 = m_3 m_1$ and $r_3 = m_1 m_2$. Now $A_2 = r_1(A_3) = m_2 m_3(A_3) = A_2$ and so $m_3(A_3) =$

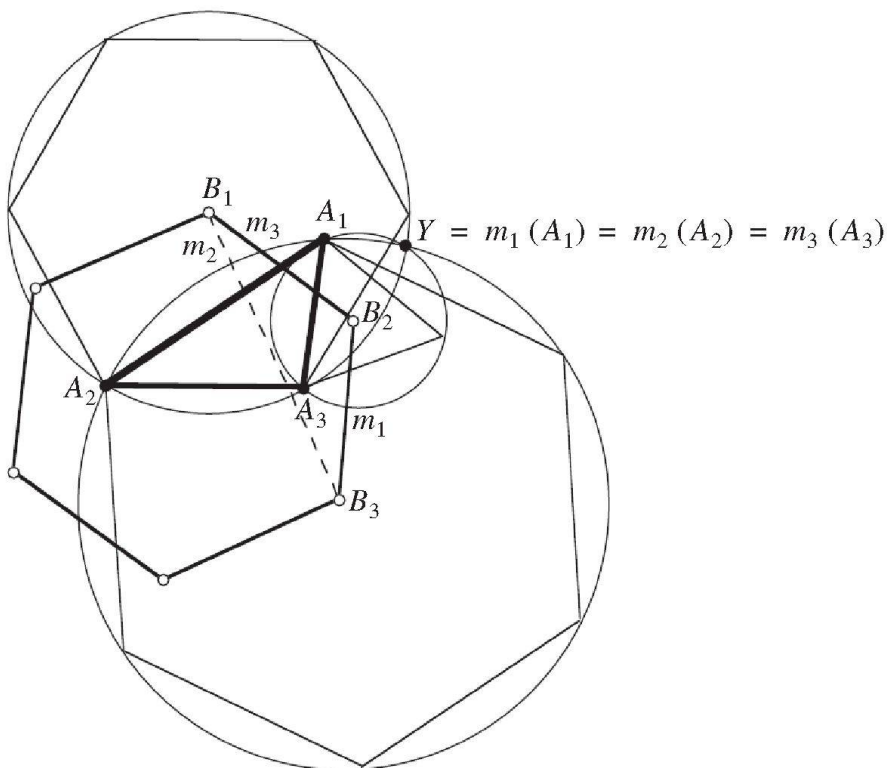
Fig. 9

$m_2^{-1}(A_2) = m_2(A_2)$. Similarly for $m_1(A_1)$, and we may put

$$m_1(A_1) = m_2(A_2) = m_3(A_3) = Y$$

where Y is some point. The fact that A_1 reflects into Y and that the line of reflection m_1 passes through B_3 shows that the circle through A_1 with centre B_3 also passes through Y. This circle is the circumcircle of the polygon centred at B_3. In an exactly similar way the circumcircles of the other two polygons with centres at B_1 and B_2 also pass through Y, and the corollary is proved. In Fig. 8 we have indicated the circumcircles and the point Y for the configuration of Fig. 6.

The results of this paper may be regarded as an extension of, and geometrical interpretation of, Schütte's theorem [3], [4]. Our proof is based on that of Stachel [5]. In [1], "dynamic proofs", that is, proofs using rotations and reflections, are used to establish several other results related to the classical Napoleon's theorem.

Acknowledgement. I am indebted to Dr. Shaun Stevens for comments on an early version of this paper.

References

[1] Finney, R.L.: Dynamic proofs of Euclidean theorems. *Math. Mag.* 43 (1970), 177–185.

[2] Martini, H.: On the theorem of Napoleon and related topics. *Math. Semesterber.* 43 (1996), 47–64.

[3] Schütte, K.: Eine Verallgemeinerung des Satzes von Napoleon. *Math. Semesterber.* 34 (1987), 256–258.

[4] Schütte, K.: Neue Fassung einer Verallgemeinerung des Satzes von Napoleon. *Elem. Math.* 44 (1989), 133–138.

[5] Stachel, H.: Zu K. Schüttes Verallgemeinerung des Satzes von Napoleon. *Elem. Math.* 46 (1991), 25–27.

G.C. Shephard
School of Mathematics
University of East Anglia
Norwich NR4 7TJ, England, U.K.
e-mail: g.c.shephard@uea.ac.uk