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Equicevian points on the altitudes of a triangle

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Let ABC be a triangle, and let a, b, c, A, B, C denote its side lengths and angles in the standard order. The letters A, B, C denote the angles, their measures, and the vertices, and the symbol AB stands for the line segment AB as well as its length and the line determined by it. When there is any ambiguity, we will talk about the point A, the line AB, the length of the line segment AB, etc. The length of the line segment AB is also denoted by $|AB|$.

For any $P \neq B$ in the plane of ABC such that BP is not parallel to AC, we let BB_P denote the cevian from B through P, and we think of B_P as undefined otherwise. Similarly, we define CC_P to be the cevian from C through P if $P \neq C$ and CP is not parallel to AB. Thus $B_P = AC \cap BP$ and $C_P = AB \cap CP$. An *A-equicevian* point is defined to be a point P through which the cevians BB_P, CC_P are equal. When talking about A-equicevian points, we often neglect the points that lie on the line BC, which are trivially A-equicevian.

Let J be any point on BC, and let X, Y be the points on the extension of AJ such that CX, BY are parallel to AB, AC, respectively; see Fig. 1. Let U, V be the points on the rays AJ, JA, respectively, that are infinitely far.

In der vorliegenden Arbeit gehen die Autoren der folgenden Fragestellung zur Dreiecksgeometrie nach: Gegeben sei ein Dreieck ABC mit einem Punkt P auf der gegebenenfalls verlängerten Höhe über der Seite BC. Es seien dann B_P bzw. C_P die Schnittpunkte der Geraden durch die Punkte B, P und A, C bzw. C, P und A, B. Die Frage besteht nun nach der Existenz von Punkten P mit der Eigenschaft, dass die Strecken BB_P und CC_P gleich lang sind. Im Gegensatz zur analogen Fragestellung, bei der P auf einer Seiten- oder Winkelhalbierenden des Dreiecks ABC liegt, fällt die Antwort im vorliegenden Fall positiv aus. Bei der Beantwortung der Frage wird man auf spezielle Polynome geführt, die unerwartete Zusammenhänge eröffnen.

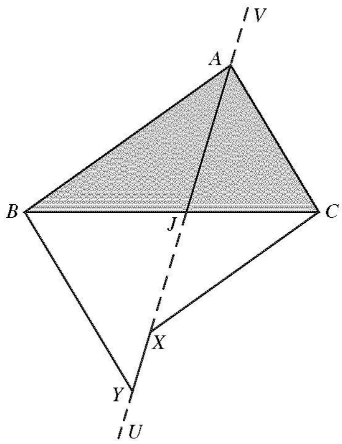

Fig. 1

It is proved in [17] and [1, Theorem 1] that if AJ is the internal angle bisector of A, and if $AB > AC$, then

(A-1) $BB_P > CC_P$ when P lies on the rays JV, YU,

(A-2) $BB_P < CC_P$ when P lies on the line segment JX,

(A-3) $BB_P = CC_P$ for at least one point P on the line segment XY.

It is also proved in [2] that if AJ is the median through A, and if $AB > AC$, then

(B-1) $BB_P > CC_P$ when P lies on the ray JV,

(B-2) $BB_P < CC_P$ when P lies on the ray JU.

In this paper, we investigate the A-equicevian points on the altitude AO from A. Theorem 2 deals with the case when $C = 90°$ and its proof is too easy to include, and Theorem 3 deals with the case when $C \neq 90°$. As a preparation, we prove a simple lemma that we shall use in the proof of Theorem 3. It is interesting to see the polynomial $X^3 + Y^3 + Z^3 - 3XYZ$, which has already appeared in the existing literature in several diverse contexts, appear in the proof of this lemma; see Remark 4. Remark 5 is concerned with another distinguished polynomial that appears in the proof of Theorem 2.

Lemma 1 *Let* $P = (x^2 - y^2 - z^2)^3 - 27x^2y^2z^2$, $R = x^{2/3} - y^{2/3} - z^{2/3}$. *Then* $P > 0$ *if and only if* $R > 0$. *Similar statements hold if the inequality sign is reversed or replaced by an equality.*

Proof. Define X, Y, Z by $X^3 = x^2$, $Y^3 = -y^2$, $Z^3 = -z^2$, and let $\omega = e^{2\pi i/3}$ be a primitive third root of 1. Since $X \geq 0$, $Y \leq 0$, $Z \leq 0$, it follows that

$$(X - Y)^2 + (Y - Z)^2 + (Z - X)^2 = 0 \Longleftrightarrow X = Y = Z = 0.$$

Also, $P = (X^3 + Y^3 + Z^3)^3 - 27X^3Y^3Z^3$. Letting $F = X^3 + Y^3 + Z^3 - 3XYZ$, we see that $P > 0 \Longleftrightarrow F > 0$ and

$$\begin{aligned} F &= (X^3 + Y^3 + Z^3) - 3XYZ \\ &= (X + Y + Z)(X + \omega Y + \omega^2 Z)(X + \omega^2 Y + \omega Z) \qquad (1) \\ &= \frac{1}{2}(X + Y + Z)\big((X - Y)^2 + (Y - Z)^2 + (Z - X)^2\big). \qquad (2) \end{aligned}$$

Therefore $P > 0 \Longleftrightarrow F > 0 \Longleftrightarrow X + Y + Z > 0 \Longleftrightarrow x^{2/3} - y^{2/3} - z^{2/3} > 0$. Similar statements hold when the sign $>$ is replaced by $<$ or by $=$. □

Theorem 2 *Let ABC be a triangle in which $C = 90°$. If $A < 45°$, then there are exactly two A-equicevian points on the line AC. One of these points lies on the side AC and the other is its reflection about BC. If $A \geq 45°$, then there are no A-equicevian points on the line AC.*

Theorem 3 *Let ABC be a triangle in which $C \neq 90°$ and $AB > AC$. Let AO be the altitude from A, and let the line drawn from C parallel to AB meet the line AO at X; see Figs. 2 and 3. Let $\mathcal{H}$ be the orthocenter of ABC, and let A^*, X^* be the reflections of A, X about BC. Let*

$$Q = \cot^{2/3} B + \cot^{2/3} C. \qquad (3)$$

(a) *There exists a unique A-equicevian point that lies on the ray OX. This point lies between X and A if C is obtuse and between X and A^* if C is acute.*

(b) *On the ray OX^*, there are no A-equicevian points, there is exactly one A-equicevian point, there are two A-equicevian points according as $Q > 1$, $Q = 1$, $Q < 1$, respectively. In the last two cases, A is necessarily less than or equal to $45°$, $|AO| \geq 8|\mathcal{H}O|$, and the A-equicevian points lie between $\mathcal{H}'$ and A, where $\mathcal{H}'$ is the point on the segment $\mathcal{H}A$ with $|\mathcal{H}'O| = 2|\mathcal{H}O|$.*

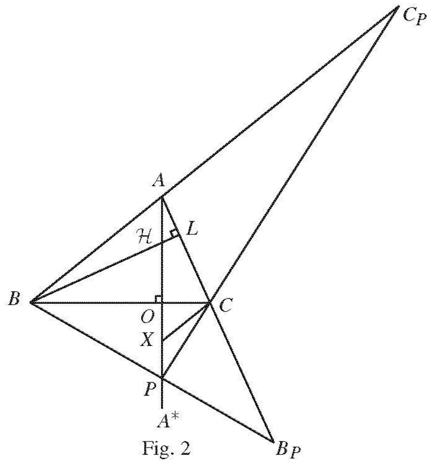

Fig. 2

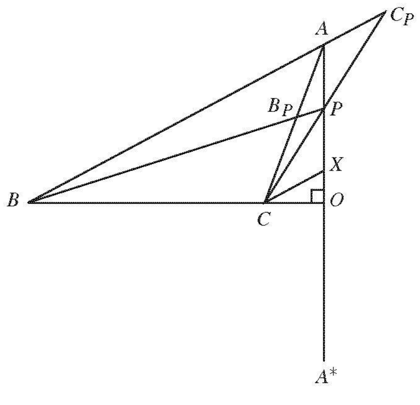

Fig. 3

Proof. We place ABC in the cartesian plane in such a way that

$$O = (0, 0), \quad A = (0, \alpha), \quad B = (-\beta, 0), \quad C = (\gamma, 0),$$

where $\alpha > 0$. Since $B < C$, it follows that $\beta > 0$ while γ is positive, zero, or negative according as C is acute, right, or obtuse, respectively. In all cases, $\beta^2 > \gamma^2$; see Figs. 2 and 3. It is easy to see that

$$A^* = (0, -\alpha), \quad \mathcal{H} = \left(0, \frac{\beta\gamma}{\alpha}\right), \quad X = \left(0, \frac{-\alpha\gamma}{\beta}\right).$$

For any point $P = (0, h)$ on the line AO, let BB_P, CC_P be the cevians through P, and let $u = BB_P$, $v = CC_P$. It is easy to find the coordinates of B_P, C_P in terms of h and then to find u, v. In fact, the equations of BB_P, AC are given, respectively, by

$$y = \frac{h}{\beta}(x + \beta), \ y = \frac{-\alpha}{\gamma}(x - \gamma).$$

Therefore

$$B_P = \left(\frac{\beta\gamma(\alpha - h)}{\gamma h + \alpha\beta}, \frac{\alpha h(\gamma + \beta)}{\gamma h + \alpha\beta}\right)$$

and

$$u^2 = (BB_P)^2 = \frac{\alpha^2(\gamma + \beta)^2(h^2 + \beta^2)}{(\gamma h + \alpha\beta)^2}.$$

By substituting $-\beta$ for γ and $-\gamma$ for β, we obtain

$$v^2 = (CC_P)^2 = \frac{\alpha^2(\gamma + \beta)^2(h^2 + \gamma^2)}{(\beta h + \alpha\gamma)^2}.$$

Subtracting v^2 from u^2 and simplifying, we obtain

$$u^2 - v^2 = \big(h^3 - (\alpha^2 - \beta^2 - \gamma^2)h + 2\alpha\beta\gamma\big)\lambda,$$

where

$$\lambda = \left(\frac{\alpha(\beta + \gamma)}{(\gamma h + \alpha\beta)(\beta h + \alpha\gamma)}\right)^2 (\beta^2 - \gamma^2)h.$$

Since $\lambda = 0$ if and only if $h = 0$, it follows that $u^2 - v^2$ vanishes if and only if $h = 0$ or $f(h) = 0$, where

$$f(T) = T^3 - (\alpha^2 - \beta^2 - \gamma^2)T + 2\alpha\beta\gamma. \tag{4}$$

We will neglect the trivial case $h = 0$; this corresponds to the point $(0, 0)$ which is trivially A-equicevian. We also let

$$E = \alpha^2 - \beta^2 - \gamma^2. \tag{5}$$

Thus $f(T) = T^3 - ET + 2\alpha\beta\gamma$.

We now consider the cases $\gamma > 0$ and $\gamma < 0$ separately.

Case 1. $\gamma > 0$ (i.e., C is acute). Consider $g(T) = -f(-T) = T^3 - ET - 2\alpha\beta\gamma$ and use Descartes' rule of signs; see [7, p. 76] and [20, p. 121]. No matter what the sign of E is, it follows that $g(T)$ has at most one positive zero. Therefore $f(T)$ has at most one negative zero. Since $f(-\infty) = -\infty < 0$ and $f(0) > 0$, it follows that f has exactly one negative zero. In fact, this negative zero lies between $-\alpha$ and $-\alpha\gamma/\beta$ because

$$\begin{aligned} f(-\alpha) &= -\alpha^3 + \alpha^3 - \alpha\beta^2 - \alpha\gamma^2 + 2\alpha\beta\gamma \\ &= -\alpha(\beta-\gamma)^2 \\ &< 0, \end{aligned}$$

$$\begin{aligned} f\left(\frac{-\alpha\gamma}{\beta}\right) &= \frac{-\alpha^3\gamma^3}{\beta^3} + \frac{\alpha^3\gamma}{\beta} - \alpha\beta\gamma - \frac{\alpha\gamma^3}{\beta} + 2\alpha\beta\gamma \\ &= \frac{\alpha\gamma}{\beta^3}(\alpha^2+\beta^2)(\beta^2-\gamma^2) \\ &> 0. \end{aligned}$$

Thus the A-equicevian point corresponding to the unique negative zero of f lies between X and the reflection $A^* = (0, -\alpha)$ of A about BC.

It remains to find the possible positive zeros. We already know that f has a unique negative zero. Therefore it has two, one, or no positive zeros if and only if $\Delta > 0$, $\Delta = 0$, or $\Delta < 0$, respectively, where Δ is the discriminant of f. The discriminant of $T^3 + pT + q$ is given by $-4p^3 - 27q^2$; see for example [7, Theorem 1, p. 46] or [8, p. 112]. Thus

$$\Delta = 4[(\alpha^2 - \beta^2 - \gamma^2)^3 - 27\alpha^2\beta^2\gamma^2]. \tag{6}$$

Hence it follows from Lemma 1 and the facts that $\beta/\alpha = \cot B$ and $\gamma/\alpha = \cot C$ that

$$\begin{aligned} f \text{ has two positive zeros} &\iff \Delta > 0 \iff Q < 1, \\ f \text{ has a unique positive zero} &\iff \Delta = 0 \iff Q = 1, \\ f \text{ has no positive zeros} &\iff \Delta < 0 \iff Q > 1, \end{aligned}$$

where Δ and Q are as given in (6) and (3).

We shall see now that if f has positive zeros, then $A \le 45°$ and in particular the orthocenter $\mathcal{H}$ is interior. Also, $8|O\mathcal{H}| \le |AO|$ and the zeros of f lie on $A\mathcal{H}'$, where $\mathcal{H}'$ is the point on $A\mathcal{H}$ with $|\mathcal{H}'O| = 2|\mathcal{H}O|$.

So suppose that f has (one or two) positive zeros. Thus $\Delta \ge 0$ and $Q \le 1$. It follows from $\Delta \ge 0$ and (6) and (5) that

$$E = \alpha^2 - \beta^2 - \gamma^2 \ \ge\ 3\,(\alpha\beta\gamma)^{2/3} \ \ge\ 0. \tag{7}$$

It also follows from $Q \le 1$ and the AM-GM inequality that

$$\left(\frac{\beta\gamma}{\alpha^2}\right)^{1/3} = \left(\frac{\beta}{\alpha}\right)^{1/3}\left(\frac{\gamma}{\alpha}\right)^{1/3} \le \frac{1}{2}\left(\left(\frac{\beta}{\alpha}\right)^{2/3} + \left(\frac{\gamma}{\alpha}\right)^{2/3}\right) \le \frac{1}{2}. \tag{8}$$

This shows that $8|\mathcal{H}O| \le |AO|$.

From $f'(T) = 3T^2 - E$, we see that f decreases for $0 < T < \sqrt{E/3}$ and increases on $T > \sqrt{E/3}$. Thus the graph of f looks like a parabola with a vertex at $\sqrt{E/3}$ such that $f(\sqrt{E/3}) \le 0$, $f(0) > 0$, $f(\infty) > 0$.

Since $f(\alpha) = \alpha(\beta+\gamma)^2 > 0$ and $f'(\alpha) = 2\alpha^2+\beta^2+\gamma^2 > 0$, it follows that α is greater than the greater zero of f.

Similarly,

$$\begin{aligned}
f\left(\frac{2\beta\gamma}{\alpha}\right) &= \left(\frac{2\beta\gamma}{\alpha}\right)^3 + \frac{2\beta\gamma}{\alpha}(\beta^2+\gamma^2-\alpha^2) + 2\alpha\beta\gamma \\
&= \frac{2\beta\gamma}{\alpha^3}(4\beta^2\gamma^2 + \alpha^2\beta^2 + \alpha^2\gamma^2) \\
&> 0,
\end{aligned}$$

$$\begin{aligned}
f'\left(\frac{2\beta\gamma}{\alpha}\right) &= 3\left(\frac{2\beta\gamma}{\alpha}\right)^2 - (\alpha^2-\beta^2-\gamma^2) \\
&\le 3\left(\frac{2\beta\gamma}{\alpha}\right)^2 - 3(\alpha\beta\gamma)^{\frac{2}{3}} \qquad \text{(by (7))} \\
&= 3\left(\frac{\beta\gamma}{\alpha}\right)^2\left(4 - \left(\frac{\alpha^2}{\beta\gamma}\right)^{\frac{4}{3}}\right) \\
&< 3\left(\frac{\beta\gamma}{\alpha}\right)^2(4-2^4) \qquad \text{(by (8))} \\
&< 0.
\end{aligned}$$

Therefore the smallest zero of f is greater than $2\beta\gamma/\alpha$. Hence the positive zeros lie between $2\beta\gamma/\alpha$ and α. In other words, if any A-equicevian points lie above BC, then they lie between $\mathcal{H}'$ and A, where $\mathcal{H}'$ is the point on $\mathcal{H}A$ such that $|\mathcal{H}'O| = 2|\mathcal{H}O|$. One can also show that the constant 2 cannot be improved.

It remains to show that the condition $\cot^{2/3} B + \cot^{2/3} C \le 1$ implies that $\cot A \ge 1$, i.e., $A \le 45°$. Let $x = \cot^{1/3} B$, $y = \cot^{1/3} C$. Then $\cot A = (1 - x^3y^3)/(x^3+y^3)$. Thus it is enough to show that the minimum of the function $g(x, y) = (1 - x^3y^3)/(x^3+y^3)$ on the region Ω defined by $h(x, y) = x^2+y^2 \le 1$, $x, y \ge 0$ is 1.

From

$$\nabla g(x, y) = \left(\frac{-3x^2\left(y^6+1\right)}{\left(x^3+y^3\right)^2}, \frac{-3y^2\left(x^6+1\right)}{\left(x^3+y^3\right)^2}\right),$$

it follows that g has no interior critical points. On the boundary lines $x = 0$ and $y = 0$, g attains its minimum at $(0, 1)$ and $(1, 0)$ and the minimum is 1. On the boundary $x^2+y^2 = 1$, we use Lagrange's multipliers to obtain

$$\frac{-3x^2\left(y^6+1\right)}{\left(x^3+y^3\right)^2} = 2\lambda x, \qquad \frac{-3y^2\left(x^6+1\right)}{\left(x^3+y^3\right)^2} = 2\lambda y.$$

Multiplying these equations by y and x, respectively, and subtracting, we obtain

$$0 = -3x^2y(y^6+1) + 3y^2x(x^6+1) = 3xy(y-x)(1+xy)(1-xy)^2.$$

Since $xy = 1$ and $x^2 + y^2 = 1$ do not intersect, we are left with the possibility $x = y = \sqrt{2}/2$ with $g(x, y) = 7\sqrt{2}/8 > 1$. Therefore the minimum of g on Ω is 1. Thus $A \le 45°$. This completes the proof of the acute case.

Case 2. $\gamma < 0$ (i.e., C is obtuse). By Descartes' rule of signs, f has at most one positive zero. Since $f(0) < 0$ and $f(\infty) > 0$, it follows that f has exactly one positive zero. As in Case 1, we have

$$f(\alpha) = \alpha(\beta-\gamma)^2 > 0,$$
$$f\left(\frac{-\alpha\gamma}{\beta}\right) = \frac{\alpha\gamma}{\beta^3}(\alpha^2+\beta^2)(\beta^2-\gamma^2) < 0.$$

Therefore the unique positive zero of f lies between α and $-\alpha\gamma/\beta$. The corresponding A-equicevian point lies between A and X. The rest is similar to the treatment of Case 1, and we skip it. □

Remark 4 The polynomial $F = x^3 + y^3 + z^3 - 3xyz$ and its wonderful factorization

$$F := x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) \qquad (9)$$

that appears in (1) have been a source of great fascination to many. In a letter to Lucy Donnelly in 1940, Bertrand Russell confessed that he used, when excited, to calm himself by reciting the three factors of $a^3 + b^3 + c^3 - 3abc$; see [6]. The very first section, Section 1.1 (pp. 3–7), of [3] is devoted to the factorization (9) and its application to other problems. The mysterious graph in [14] is the graph of a simple deformation of F and it is the factorization (9) that is used in [11] to remove this mystery. This same factorization (9) is what the Putnam problem (B-1) of [21] is about. Also, the polynomial F is *the favourite polynomial* referred to in the title of [16], where the author collects together properties of this polynomial and applications of its factorization (9). Also, the factorization (9) is used in [19] to obtain an immediate elegant derivation of Cardan's formula for the roots of a cubic, and in [3] to give a proof of the AM-GM inequality in three variables (which follows immediately from (2)). In [9, Example, p. 60], it is observed that if x, y, z are complex numbers located in the complex plane, then the triangle (x, y, z) is equilateral if and only if $(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) = 0$. In other words, $x^3+y^3+z^3-3xyz = 0$ if and only if the triangle (x, y, z) is equilateral or has the origin as its centroid.

The factorization (9) has also played a crucial role in the birth of the theory of *Group Representations*; see [15] and [5]. Quoting from [6], *this factorization was, historically, the seed that, watered by Frobenius, grew into the great subject of group representation theory*. In technical terms, the determinant of the cyclic group $\mathbb{Z}_3$, being the circulant matrix with row $[x \;\; y \;\; z]$, is given by

$$\det \mathbb{Z}_3 = x^3 + y^3 + z^3 - 3xyz, \qquad (10)$$

and the fact that it factors into linear factors as in (9) is a manifestation of $\mathbb{Z}_3$ being abelian. To explain, given any finite group $G = \{x, y, z, \ldots\}$, we may think of its elements as indeterminates and of its multiplication table (with the identity element all over the main diagonal) as a matrix. Then the determinant of this matrix, a polynomial of degree n in n indeterminates, is known as the determinant of G. A wonderful theorem that combines works of Dedekind, Burnside, and Frobenius states that G is abelian if and only if its determinant factors into linear factors. In view of (10) and the factorization in (9), the Dedekind-Burnside-Frobenius theorem immediately yields a proof, a truly hilarious proof indeed, that $\mathbb{Z}_3$ is abelian.

Finally, many of the beautiful surprises that abound in the literature on Hilbert's seventeenth problem seem to be related to the polynomial $F = x^3 + y^3 + z^3 - 3xyz$. The first example of a positive definite polynomial which is not a sum of squares of polynomials is the polynomial

$$M(X, Y, Z) = Z^6 + X^4Y^2 + X^2Y^4 - 3X^2Y^2Z^2$$

(or its dehomogenization $M_*(X, Y) = 1+X^4Y^2+X^2Y^4-3X^2Y^2$) discovered by Motzkin; see [18, p. 73]. Its relation to F is transparent and is given by

$$M\left(\frac{y^2}{x}, \frac{x^2}{y}, z\right) = F(x^2, y^2, z^2).$$

Actually, M_* is a minimal example, both in degree and in number of variables. Amazingly also, M_* can be expressed as a sum of squares of rational functions; see [18, p. 47]. Similar statements hold for the Robinson polynomial given by $R(x, y, z) = X^4Y^2 + Y^4Z^2 + Z^4X^2 - 3X^2Y^2Z^2$.

Remark 5 The polynomial $f(T) = T^3 - (\alpha^2 - \beta^2 - \gamma^2)T + 2\alpha\beta\gamma$ appearing in (4) is also very interesting and it was a pleasant surprise to us to see it come up in the context of equicevian points. If one makes the (seemingly meaningless) substitution $T = d$, $\alpha = a$, $\beta = \mathrm{i}b$, and $\gamma = \mathrm{i}c$, where $\mathrm{i} = \sqrt{-1}$, then one obtains the polynomial

$$G(a, b, c; d) = d^3 - (a^2 + b^2 + c^2)d - 2abc.$$

This is the key polynomial in [4], [11], [12], and [13] and has already shown up in so many diverse contexts as explained in these references. As detailed in [12], this polynomial (in d) appears in the context of the fencing problem for triangles, where we are to build, using a fixed amount of money, the largest triangular fence whose sides cost a, b, c units of money per unit length. It appears again in the fencing problem for quadrilaterals with costs a, b, c, and d. It also comes up in finding the largest quadrilateral with three given sides a, b, and c, and again in finding the diameter of the circle that circumscribes a quadrilateral three of whose sides have lengths a, b, and c and whose fourth side is a diameter. It also appears when trying to recover the side lengths of a triangle given the lengths of its angle bisectors. Now its twin, appearing in (4), is relevant in the totally different context of finding the equicevian points on the altitudes of a given triangle.

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