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On some results related to Napoleon configurations

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1 Introduction

In this short article we discuss some results from planar Euclidean geometry which have a close connection to Napoleon's theorem. They are summarized in Theorem 1. The statement of Theorem 1 appears in [1] where the proof is based on coordinate descriptions and algebraic computations. Since both Theorem 1 and Napoleon's theorem (see Theorem 2) are elementary geometric results, it makes sense to provide a proof that remains in the same simple geometric domain. For that reason, the arguments presented in the current paper are entirely in the spirit of synthetic Euclidean geometry and use only geometric methods with almost no algebraic computations. Thus, one gets a better feeling for the geometry and the properties of Napoleon configurations.

Definition 1. *Let $\triangle ABC$ be an arbitrary triangle. We say that the points A_1, B_1 and C_1 form a non-overlapping Napoleon configuration for the triangle $\triangle ABC$ if all three triangles $\triangle ABC_1$, $\triangle AB_1C$ and $\triangle A_1BC$ are equilateral and no one of them overlaps with $\triangle ABC$ (see Figure 1). Alternatively, we say that the points A_1', B_1' and C_1' form an overlapping Napoleon configuration for $\triangle ABC$ if all three triangles $\triangle ABC_1'$, $\triangle AB_1'C$ and $\triangle A_1'BC$ are equilateral and all of them overlap with $\triangle ABC$.*

Um den Satz von Napoleon kreisen in der Euklidischen Geometrie zahlreiche Varianten. Bekannt sind etwa die Kiepert-Dreiecke und deren schöne Eigenschaften. Branko Grünbaum hat 2001 eine besonders ausführliche Version des Satzes von Napoleon formuliert, in der zahlreiche neue Eigenschaften der Konfiguration beschrieben werden. Grünbaum benutzt in seinem Beweis Methoden der analytischen Geometrie. Der Autor der vorliegenden Arbeit beweist nun Grünbaums Variante des Satzes mit elementaren Methoden der synthetischen Geometrie, die sich darüberhinaus als besonders anschaulich erweisen.

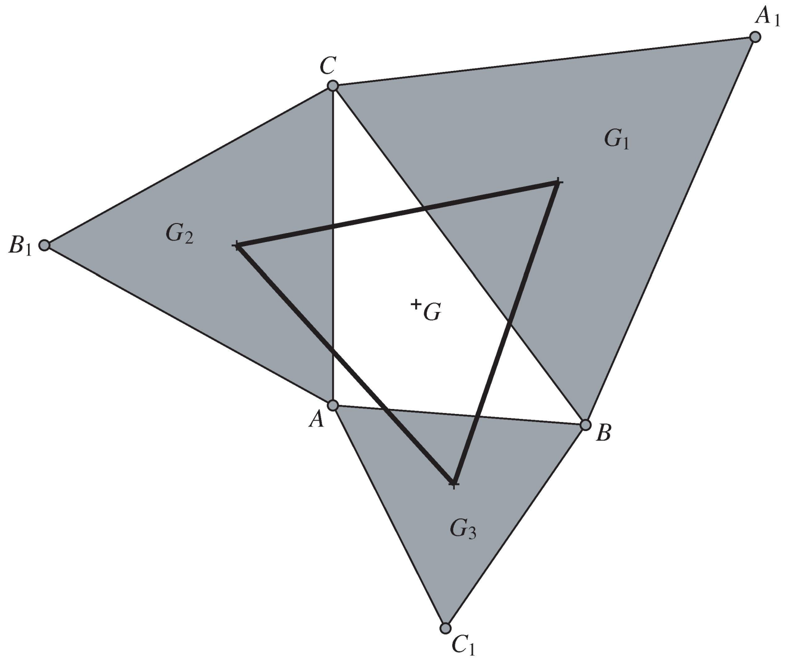

Fig. 1 A non-overlapping Napoleon configuration and the first part of Napoleon's theorem.

2 The Main Result

The main result of the current article is the following theorem.

Theorem 1. *Let us have an arbitrary triangle $\triangle ABC$ and let A_1, B_1 and C_1 form a non-overlapping Napoleon configuration for that triangle. Denote the midpoints of B_1C_1, C_1A_1 and A_1B_1 by A_2, B_2 and C_2 respectively. Also, denote the centroids of the triangles $\triangle A_1BC$, $\triangle AB_1C$ and $\triangle ABC_1$ by G_1, G_2 and G_3 respectively. Then the following statements are true:*

1. *The triangles $\triangle A_2B_2C$, $\triangle AB_2C_2$ and $\triangle A_2BC_2$ are equilateral;*
2. *The centroids A^*, B^*, C^* of $\triangle AB_2C_2$, $\triangle A_2B_2C$, $\triangle A_2BC_2$ respectively are vertices of an equilateral triangle, whose centroid coincides with the centroid G of $\triangle ABC$;*

Similarly, let A_1', B_1' and C_1' be an overlapping Napoleon configuration for $\triangle ABC$. Denote the midpoints of $B_1'C_1'$, $C_1'A_1'$ and $A_1'B_1'$ by A_2', B_2' and C_2' respectively. Also, denote the centroids of triangles $\triangle A_1'BC$, $\triangle AB_1'C$ and $\triangle ABC_1'$ by G_1', G_2' and G_3' respectively. Then

3. *The triangles $\triangle A_2'B_2'C$, $\triangle AB_2'C_2'$ and $\triangle A_2'BC_2'$ are equilateral;*
4. *The centroids A^{**}, B^{**}, C^{**} of $\triangle AB_2'C_2'$, $\triangle A_2'BC_2'$, $\triangle A_2'B_2'C$ respectively are vertices of an equilateral triangle, whose centroid coincides with the centroid G of $\triangle ABC$;*

5. *Triangle* $\triangle A^*B^*C^*$ *is homothetic to the triangle* $\triangle G_1'G_2'G_3'$ *with homothetic center* G *and a coefficient of similarity* $-1/2$;
6. *Triangle* $\triangle A^{**}B^{**}C^{**}$ *is homothetic to the triangle* $\triangle G_1G_2G_3$ *with homothetic center* G *and a coefficient of similarity* $-1/2$.
7. *The area of* $\triangle ABC$ *equals four times the algebraic sum of the areas of* $\triangle A^*B^*C^*$ *and* $\triangle A^{**}B^{**}C^{**}$.

3 Napoleon's Theorem

Napoleon's theorem is a beautiful result from planar Euclidean geometry and there are various ways to prove it. In order to make this article more self-contained, we present one possible geometrically oriented proof. Before we state and prove Napoleon's theorem we are going to need the following lemma.

Lemma 1. *Given an arbitrary triangle* $\triangle ABC$, *let* A_1, B_1 *and* C_1 *form a non-overlapping Napoleon configuration for that triangle. Then, the following properties are true:*

1. *The segments* AA_1, BB_1 *and* CC_1 *are of equal length. In other words,* $AA_1 = BB_1 = CC_1$;
2. *They intersect at a common point, denoted by* J;
3. $\measuredangle AJB = \measuredangle BJC = \measuredangle CJA = 120°$;
4. *The circles* K_1, K_2 *and* K_3 *circumscribed around the equilateral triangles* $\triangle A_1BC$, $\triangle AB_1C$ *and* $\triangle ABC_1$ *respectively pass through the point* J *(see Figure 2).*

Proof. Perform a 60° rotation R_A around the point A in counterclockwise direction. Since $AC = AB_1$ and $\measuredangle CAB_1 = 60°$, the point C is mapped to the point B_1. Similarly, C_1 is mapped to B. Therefore the segment CC_1 maps to the segment B_1B. This implies that $BB_1 = CC_1$ (see Figure 2). Moreover, if we denote by J the intersection point of BB_1 and CC_1, then $\measuredangle CJB_1 = \measuredangle C_1JB = 60°$ and $\measuredangle BJC = 180° - \measuredangle CJB_1 = 180° - 60° = 120°$. We are going to show that the points A, J and A_1 lie on the same line.

Notice that $\measuredangle CJB_1 = \measuredangle CAB_1 = 60°$. Therefore the quadrilateral CB_1AJ is inscribed in a circle K_2. Then, $\measuredangle CJA = 180° - \measuredangle AB_1C = 180° - 60° = 120°$. Since $\measuredangle BJC + \measuredangle CA_1B = 120° + 60° = 180°$, the points B, A_1, C and J lie on a circle K_1. From here we can conclude that $\measuredangle A_1JC = \measuredangle A_1BC = 60°$. Then, $\measuredangle A_1JA = \measuredangle A_1JC + \measuredangle CJA = 60° + 120° = 180°$. That means that J belongs to the straight line AA_1.

If we perform another 60° counterclockwise rotation R_B, this time around the point B, it will turn out that AA_1 is mapped to C_1C. Therefore, $AA_1 = CC_1$. Also, $\measuredangle AJB = 360° - \measuredangle BJC - \measuredangle CJA = 360° - 120° - 120° = 120°$. Since $\measuredangle AJB + \measuredangle BC_1A = 120° + 60° = 180°$, the points A, C_1, B and J lie on a circle K_3. We see that the circles K_1, K_2, K_3 all pass through the same point J. This completes the proof of Lemma 1. □

Remark. The point J from Lemma 1 (see also Figure 2) is often called Fermat point or alternatively Torricelli point.

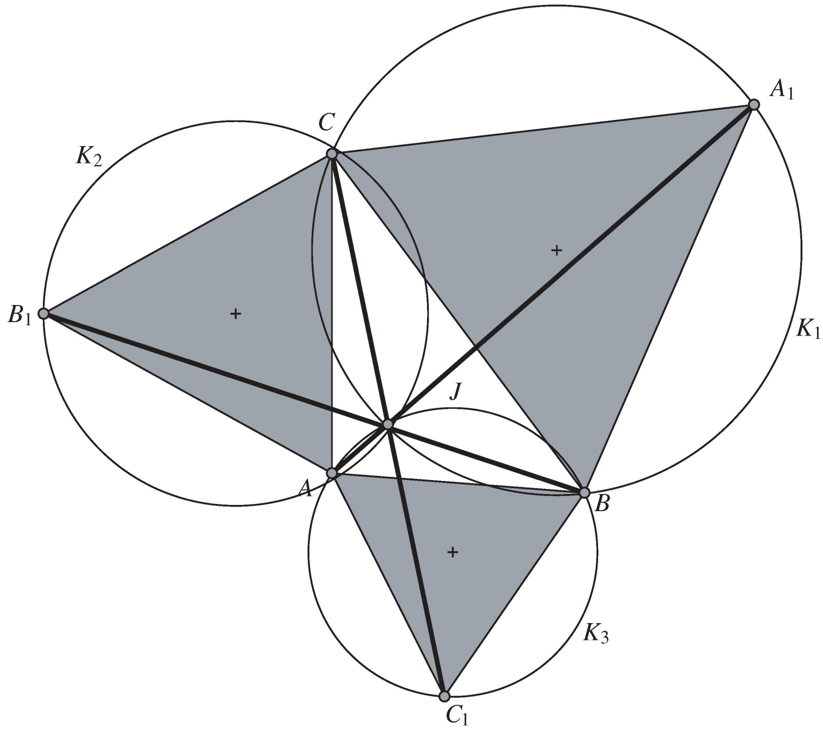

Fig. 2 Constructions in the proof of Lemma 1.

Next, we are ready to state and prove Napoleon's theorem.

Theorem 2. *Let $\triangle ABC$ be an arbitrary triangle and let G be its centroid. Then, the following statements are true:*

1. *Assume A_1, B_1 and C_1 form a non-overlapping Napoleon configuration for that triangle. Denote the centroids of the triangles $\triangle A_1BC$, $\triangle AB_1C$ and $\triangle ABC_1$ by G_1, G_2 and G_3 respectively. Then, the triangle $\triangle G_1G_2G_3$ is equilateral with a centroid coinciding with the point G;*
2. *Let A_1', B_1' and C_1' form an overlapping Napoleon configuration for that triangle. Denote the centroids of triangles $\triangle A_1'BC$, $\triangle AB_1'C$ and $\triangle ABC_1'$ by G_1', G_2' and G_3' respectively. Then, the triangle $\triangle G_1'G_2'G_3'$ is equilateral with a centroid coinciding with the point G;*
3. *The area of $\triangle ABC$ equals the algebraic sum of the areas of $\triangle G_1G_2G_3$ and $\triangle G_1'G_2'G_3'$.*

Proof. We start with the first claim of the theorem (see also Figure 3). Let M_1, M_2 and M_3 be the midpoints of the edges BC, CA and AB respectively. Since G is the centroid

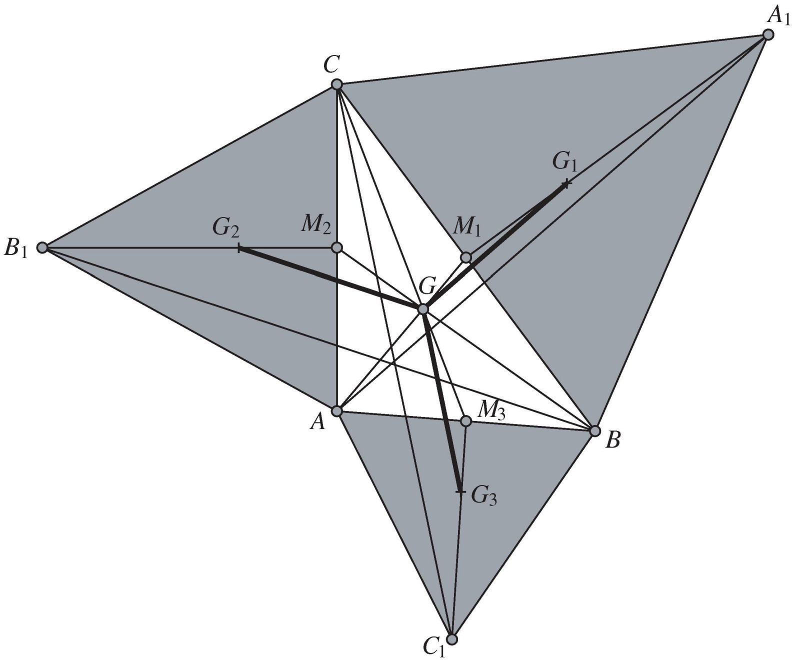

Fig. 3 Constructions in the proof of Napoleon's theorem.

of $\triangle ABC$ and G_2 is the centroid of $\triangle AB_1C$, we have the ratios $M_2G : M_2B = M_2G_2 : M_2B_1 = 1 : 3$. Therefore, by the intercept theorem $GG_2 = \frac{1}{3}BB_1$ and GG_2 is parallel to BB_1. Analogously, $GG_1 = \frac{1}{3}AA_1$, GG_1 is parallel to AA_1, $GG_3 = \frac{1}{3}CC_1$ and GG_3 is parallel to CC_1. By part 1 of Lemma 1, $AA_1 = BB_1 = CC_1$, hence $GG_1 = GG_2 = GG_3$. By part 3 of Lemma 1, $\measuredangle AJB = \measuredangle BJC = \measuredangle CJA = 120°$, so $\measuredangle G_1GG_2 = \measuredangle G_2GG_3 = \measuredangle G_3GG_1 = 120°$.

We can conclude form here that $\triangle G_1G_2G \cong \triangle G_2G_3G \cong \triangle G_3G_1G$ and hence $G_1G_2 = G_2G_3 = G_3G_1$, that is, the triangle $\triangle G_1G_2G_3$ is equilateral.

The proof of claim 2 from Napoleon's theorem is analogous to the proof of claim 1. We just have to consider overlapping configurations and rename their notations appropriately.

In order to prove claim 3 form Theorem 2, we are going to show that $\text{Area}(\triangle G_1G_2G_3) = \frac{1}{2}\,\text{Area}(\triangle ABC) + \frac{1}{6}(\text{Area}(\triangle A_1BC) + \text{Area}(\triangle AB_1C) + \text{Area}(\triangle ABC_1))$. Let point P be the reflection image of the vertex C with respect to the line G_1G_2. In other words, P is chosen so that G_1G_2 is the perpendicular bisector of CP. Hence, $\triangle G_1G_2C \cong \triangle G_1G_2P$ and $G_2P = G_2C = G_2A$. If we denote $\measuredangle G_1G_2C = \alpha$ then $\measuredangle PG_2G_1 = \alpha$. On the one hand, $\measuredangle AG_2P = \measuredangle AG_2C - \measuredangle PG_2C = 120° - \measuredangle PG_2C = 120° - (\measuredangle PG_2G_1 + \measuredangle G_1G_2C) = 120° - 2\alpha$. On the other hand, $\measuredangle G_3G_2P = \measuredangle G_3G_2G_1 - \measuredangle PG_2G_1 = 60° - \alpha$. Therefore, $\measuredangle AG_2G_3 = \measuredangle AG_2P - \measuredangle G_3G_2P = 120° - 2\alpha - (60° - \alpha) = 60° - \alpha$. Since $G_2P = G_2A$ and $\measuredangle AG_2G_3 = \measuredangle G_3G_2P = 60° - \alpha$, the line G_2G_3 is the bisector of $\measuredangle AG_2P$ in the isosceles triangle $\triangle AG_2P$, and hence it is the perpendicular

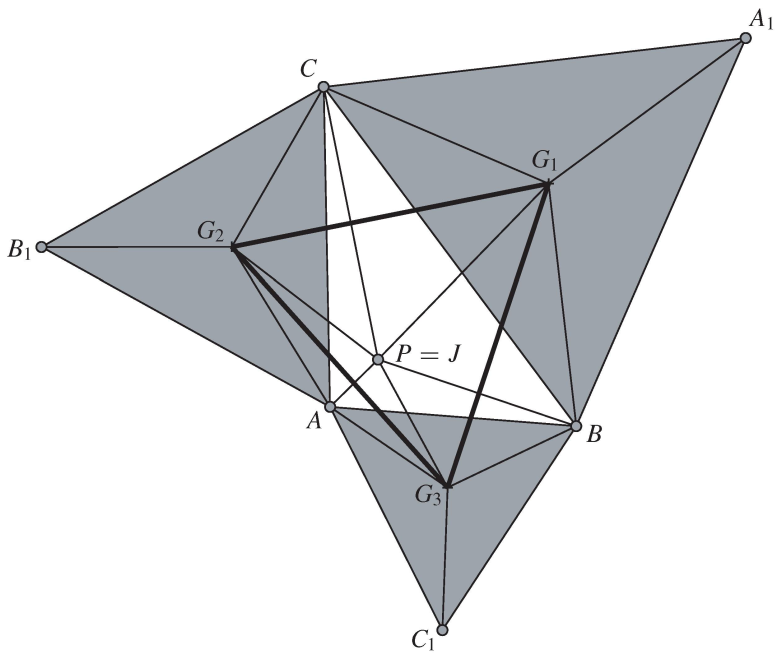

Fig. 4 Constructions in the proof of Napoleon's theorem.

bisector of the segment AP. Therefore, P is the reflection image of A with respect to G_2G_3 and $\triangle G_2G_3A \cong \triangle G_2G_3P$. Analogously, we can show that the reflection of B with respect to G_3G_1 is again P and $\triangle G_3G_1B \cong \triangle G_3G_1P$. All of the arguments above lead to the conclusion that $\text{Area}(\triangle G_1G_2G_3) = \text{Area}(\triangle G_1G_2P) + \text{Area}(\triangle G_2G_3P) + \text{Area}(\triangle G_3G_1P) = \text{Area}(\triangle G_1G_2C) + \text{Area}(\triangle G_2G_3A) + \text{Area}(\triangle G_3G_1B)$, so

$$\text{Area}(\triangle G_1G_2G_3) = \frac{1}{2}\,\text{Area}(AG_3BG_1CG_2).$$

Notice that $\text{Area}(AG_3BG_1CG_2) = \text{Area}(\triangle ABC) + \text{Area}(\triangle AG_3B) + \text{Area}(\triangle BG_1C) + \text{Area}(\triangle CB_2A) = \text{Area}(\triangle ABC) + \frac{1}{3}(\text{Area}(\triangle A_1BC) + \text{Area}(\triangle AB_1C) + \text{Area}(\triangle ABC_1))$. It follows from here that $\text{Area}(\triangle G_1G_2G_3) = \frac{1}{2}\,\text{Area}(\triangle ABC) + \frac{1}{6}(\text{Area}(\triangle A_1BC) + \text{Area}(\triangle AB_1C) + \text{Area}(\triangle ABC_1))$.

Using analogous arguments, one can show that $\text{Area}(\triangle G_1'G_2'G_3') = \frac{1}{2}\,\text{Area}(\triangle ABC) - \frac{1}{6}(\text{Area}(\triangle A_1'BC) + \text{Area}(\triangle AB_1'C) + \text{Area}(\triangle ABC_1'))$. Now, we can deduce that

$$\text{Area}(\triangle G_1G_2G_3) + \text{Area}(\triangle G_1'G_2'G_3') = \text{Area}(\triangle ABC).$$

An additional observation is that $G_1P = G_1B = G_1C = G_1A_1$ and therefore P lies on the circle K_1, circumscribed around $\triangle A_1BC$ (see Lemma 1 and Figure 2). Similarly, P lies on the circles K_2 and K_3 circumscribed around $\triangle AB_1C$ and $\triangle ABC_1$ respectively. That implies that P is the intersection point of K_1, K_2 and K_3, which was already denoted by J, i.e., $P \equiv J$. □

4 Proof of Theorem 1

This section contains the proof of the main result, namely Theorem 1. To prove this statement we are going to use several lemmas and corollaries which together will give us the desired result.

The next lemma is essentially the proof of fact 1 from Theorem 1.

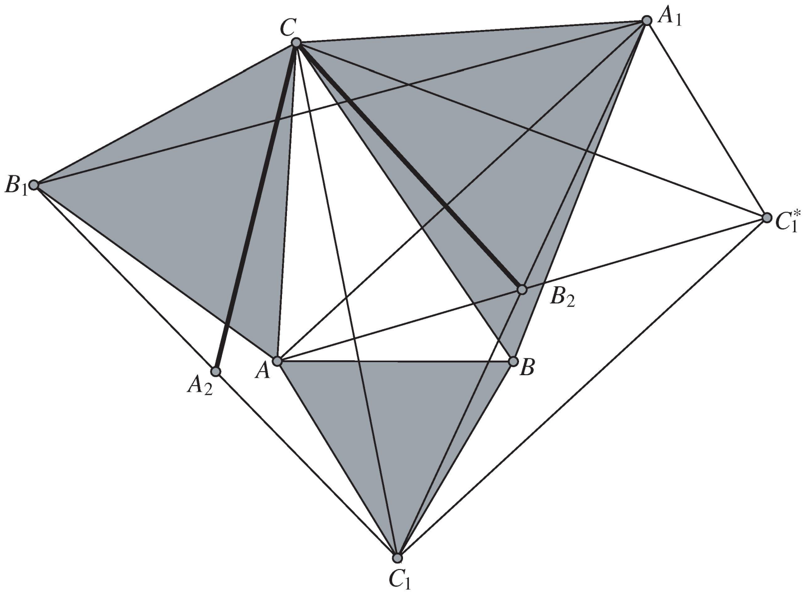

Fig. 5 Constructions in the proof of Lemma 2.

Lemma 2. *In the setting of Theorem 1, the points A_2, B_2 and C form an equilateral triangle (see Figure 5).*

Proof. Consider a 60° rotation R_C around the point C in counterclockwise direction. The point B_1 maps to A. Denote by C_1^* the image of the point C_1 (Figure 5). Then, B_1C_1 maps to AC_1^*. We are going to show that the point B_2 is the image of A_2 under the rotation R_C. Since the midpoint A_2 of B_1C_1 maps to the midpoint of the image AC_1^*, we need to prove that B_2 lies on AC_1^* and is the midpoint of that segment.

By the properties of the rotation R_C, we have that $CC_1 = CC_1^*$ and $\measuredangle C_1CC_1^* = 60°$. Therefore triangle $\triangle CC_1C_1^*$ is equilateral and so by Lemma 1 we can deduce that $C_1C_1^* = CC_1 = AA_1$.

Notice that the point A_1 is the image of B under the rotation R_C. Since C_1 maps to C_1^* we have that BC_1 maps to $A_1C_1^*$. Thus, $A_1C_1^* = BC_1 = AC_1$.

The facts that $CC_1 = AA_1$ and $A_1C_1^* = AC_1$ imply that the quadrilateral $AA_1C_1^*C_1$ is a parallelogram. For any parallelogram, the intersection point of the diagonals is the midpoint for both diagonals. That means that the midpoint B_2 of the diagonal C_1A_1 lies on the diagonal AC_1^* and is the midpoint of AC_1^*. Therefore, B_2 is the image of A_2 under the rotation R_C. Hence, $CA_2 = CB_2$ and $\measuredangle A_2CB_2 = 60°$, i.e., the triangle $\triangle A_2B_2C$ is equilateral. □

We are going to need the following intermediate statement.

Lemma 3. *Consider the equilateral triangle $\triangle ABC_1'$, overlapping $\triangle ABC$. Then, the midpoint C_2 of the segment A_1B_1 is also the midpoint of CC_1' (see Figure 6).*

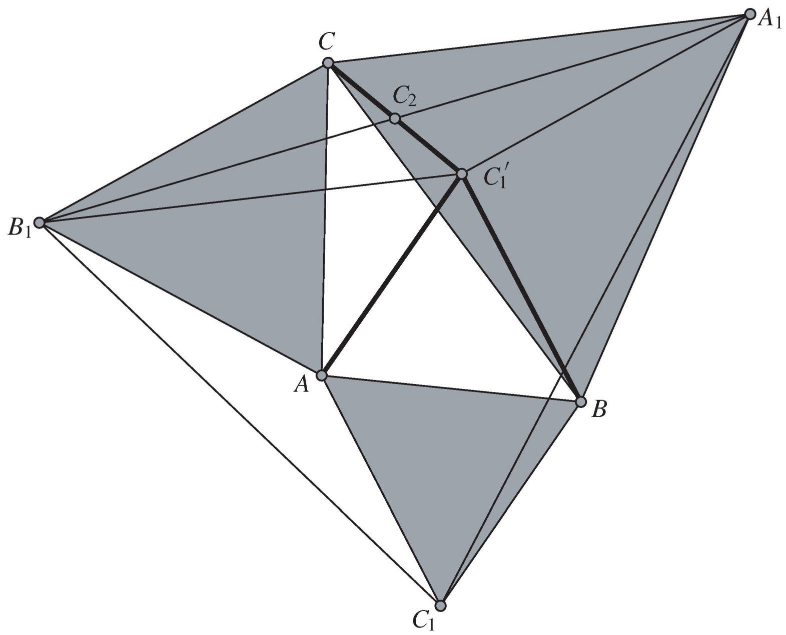

Fig. 6 Constructions in the proof of Lemma 3.

Proof. Consider a 60° degree clockwise rotation around the point A. Then B maps to C_1' and C maps to B_1. Therefore the segment BC maps to the segment $C_1'B_1$, so $BC = C_1'B_1$. Now consider a 60° degree counter-clockwise rotation around the point B. In this case A maps to C_1' and C maps to A_1. Thus, the segment AC maps to $C_1'A_1$, so $AC = C_1'A_1$. From the two identities $BC = C_1'B_1$ and $AC = C_1'A_1$ it can be concluded that the quadrilateral $B_1C_1'A_1C$ is a parallelogram. Therefore, the midpoint C_2 of the diagonal A_1B_1 is also the midpoint of the diagonal CC_1'. □

Next, we are going to locate the centroids of $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$.

Lemma 4. *The centroids of $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ coincide with the centroid G of $\triangle ABC$ (see Figure 7).*

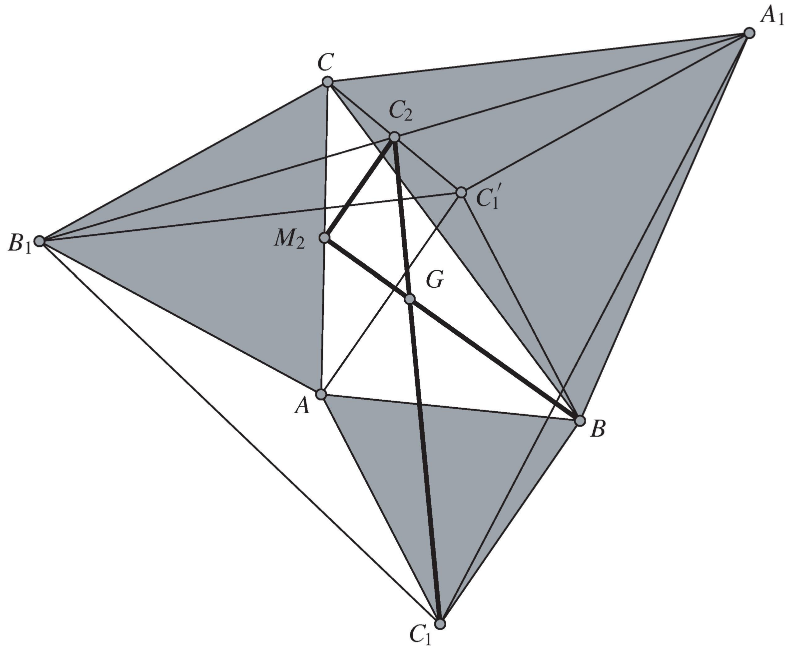

Fig. 7 Constructions in the proof of Lemma 4.

Proof. Let M_2 be the midpoint of AC. Then M_2C_2 is a mid-segment of the triangle $\triangle AC_1'C$. Therefore, M_2C_2 is parallel to AC_1' and $2M_2C_2 = AC_1'$. Since triangles $\triangle ABC_1$ and $\triangle ABC_1'$ are equilateral, the quadrilateral AC_1BC_1' is a rhombus, so $AC_1' = C_1B$ and AC_1' is parallel to C_1B. Hence M_2C_2 is parallel to C_1B and $2M_2C_2 = C_1B$. Let G' be the intersection point of BM_2 and C_1C_2. From here we can deduce that $BG' : G'M_2 = C_1G' : G'C_2 = BC_1 : C_2M_2 = 2 : 1$. But for the centroid G of $\triangle ABC$ it is true that $BG : GM_2 = 2 : 1$, so $G \equiv G'$ and G is the centroid of $\triangle A_1B_1C_1$. Since the triangles $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ have a common centroid, the statement is proved. □

The following corollary proves statements 1 and 2 from Theorem 1.

Corollary 1. *The points A, B and C form an overlapping Napoleon configuration for $\triangle A_2B_2C_2$. Moreover, the centroids A^*, B^*, C^* of the equilateral triangles $\triangle AB_2C_2$, $\triangle A_2BC_2$ and $\triangle A_2B_2C$ respectively form an equilateral triangle, whose centroid coincides with the centroid G of $\triangle ABC$.*

Proof. By Lemma 2, first applied to the triple A, B_2, C_2, then to the triple A_2, B, C_2, and finally to the triple A_2, B_2, C, we obtain the first statement of Corollary 1. Thus, the points A, B and C form an overlapping Napoleon configuration for $\triangle A_2B_2C_2$. By the classical Napoleon's theorem for overlapping configurations, it follows that the centroids A^*, B^*, C^* of $\triangle AB_2C_2$, $\triangle A_2BC_2$ and $\triangle A_2B_2C$ respectively form an equilateral triangle whose centroid coincides with the centroid of $\triangle A_2B_2C_2$. By Lemma 4, the centroid of $\triangle A_2B_2C_2$ coincides with the centroid G of $\triangle ABC$. The corollary is proved. □

Notice that the proof of the statements 3 and 4 from Theorem 1 is absolutely analogous to the proof of the statements 1 and 2. All we have to do is to follow more or less the same arguments, just changing the notation appropriately. What is left is the verification of the last three claims from Theorem 1. We proceed with the following lemma:

Lemma 5. *Consider the centroids C^* and G_3' of the equilateral triangles $\triangle A_2B_2C$ and $\triangle ABC_1'$ respectively. Then G_3' maps to C^* under a homothetic transformation of dilation factor $-1/2$ with respect to the centroid G of $\triangle ABC$ (see Figure 8).*

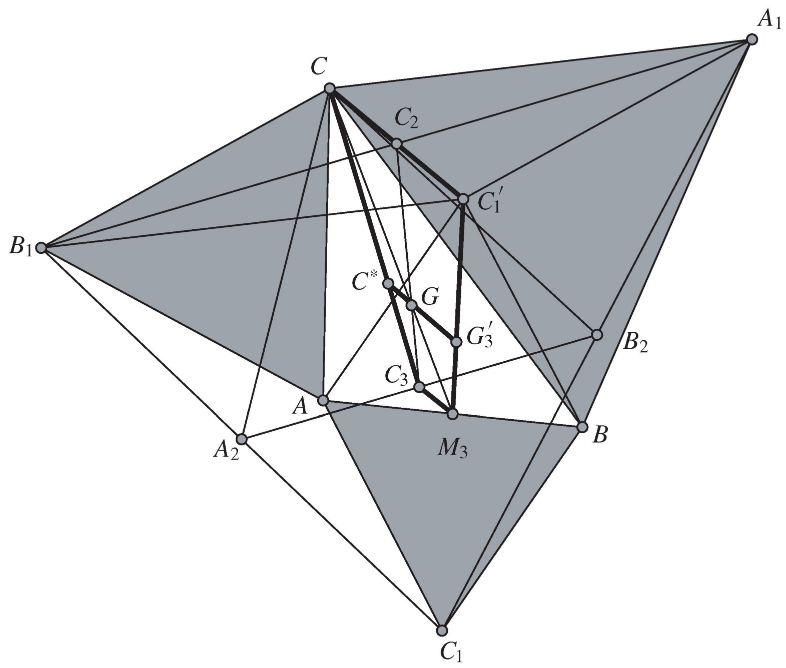

Fig. 8 Constructions in the proof of Lemma 5.

Proof. Perform a homothetic transformation of dilation factor $-1/2$ with respect to the centroid G of $\triangle ABC$. By Lemma 4 the point G is also the centroid of $\triangle A_1B_1C_1$. Then A_1B_1 maps to A_2B_2 and so the midpoint C_2 of A_1B_1 maps to the midpoint C_3 of A_2B_2. Also, the vertex C maps to the midpoint M_3 of AB because G is the centroid of $\triangle ABC$ (see Figure 8). From here we can conclude that $C_3G : GC_2 = 1 : 2$ and $M_3G : GC = 1 : 2$ which transforms into $C_3G : C_3C_2 = 1 : 3$ and $M_3G : M_3C = 1 : 3$. As C^* is the centroid of $\triangle A_2B_2C$, we can see that $C_3C^* : C_3C = C_3G : C_3C_2 = C^*G : CC_2 = 1 : 3$ and C^*G is parallel to CC_2. Similarly, G_3' is the centroid of $\triangle ABC_1'$, so $M_3G_3' : M_3C_1' =$

$M_3G : M_3C = G_3'G : C_1'C = 1 : 3$ and $G_3'G$ is parallel to $C_1'C$. By Lemma 3, C_2 is the midpoint of CC_1' which means that both GC^* and GG_3' are parallel to the same line CC_2. Therefore G belongs to C^*G_3'. Moreover, $C^*G = \frac{1}{3}CC_2 = \frac{1}{6}CC_1'$ and $G_3'G = \frac{1}{3}CC_1'$. Hence $C^*G : GG_3' = 1 : 2$, so the point C^* is the image of the point G_3' under the homothetic transformation of factor $-1/2$ with respect to G. □

After establishing the previous result, we are ready to confirm the validity of statements 5, 6 and 7 from Theorem 1.

Corollary 2. *In the setting of Theorem 1, triangle $\triangle A^*B^*C^*$ is homothetic to the triangle $\triangle G_1'G_2'G_3'$ with a homothetic center G and a coefficient of similarity $-1/2$. Similarly, triangle $\triangle A^{**}B^{**}C^{**}$ is homothetic to the triangle $\triangle G_1G_2G_3$ with a homothetic center G and a coefficient of similarity $-1/2$. Moreover, the area of $\triangle ABC$ equals four times the algebraic sum of the areas of $\triangle A^*B^*C^*$ and $\triangle A^{**}B^{**}C^{**}$.*

Proof. Applying Lemma 5 first to the pair of centroids C^* and G_3' of the equilateral triangles $\triangle A_2B_2C$ and $\triangle ABC_1'$, then to the centroids A^* and G_1' of the equilateral triangles $\triangle AB_2C_2$ and $\triangle A_1'BC$, and finally to the centroids B^* and G_2' of the equilateral triangles $\triangle A_2BC_2$ and $\triangle AB_1'C$, we conclude that triangle $\triangle A^*B^*C^*$ is homothetic to the triangle $\triangle G_1'G_2'G_3'$ with respect to G and a dilation coefficient $-1/2$. Analogously, the same is true for the equilateral triangles $\triangle A^{**}B^{**}C^{**}$ and $\triangle G_1G_2G_3$. Finally, due to the homothety, the area of $\triangle A^*B^*C^*$ is $1/4$ of the area of $\triangle G_1'G_2'G_3'$ and the area of $\triangle A^{**}B^{**}C^{**}$ is $1/4$ of the area of $\triangle G_1G_2G_3$. Since by Napoleon's theorem the area of $\triangle ABC$ equals the algebraic sum of the areas of $\triangle G_1G_2G_3$ and $\triangle G_1'G_2'G_3'$, we conclude that the area of $\triangle ABC$ equals four times the algebraic sum of the areas of $\triangle A^*B^*C^*$ and $\triangle A^{**}B^{**}C^{**}$. This completes the proof of the corollary. □

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