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Contractible Hamiltonian cycles in triangulated surfaces

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1 Introduction

By a graph $G := (V, E)$, we mean a simple graph with a vertex set V and an edge set E. Thus, G does not have any loops or double edges. A *surface* will always mean a compact, connected two-dimensional orientable manifold without a boundary. A *map* on a surface S is an embedding of a graph G with a finite number of vertices such that the components, which are called *faces* of $S \setminus G$, are topological 2-cells. Hence, the closure of a component in $S \setminus G$ is a $p-$gonal disk, i.e., a 2-disk whose boundary is a $p-$gon for some integer $p \geq 3$. We call G the edge graph of the map and the vertices and edges of G are called the vertices and edges of the map.

A map is called $\{p, q\}$-*equivelar*, $p, q \geq 3$, if each face of the map is a p-gonal 2-disk and each vertex is incident with exactly q faces. If $p = 3$, the map is called a q-equivelar

Triangulationen von geschlossenen Flächen sind nicht nur aus der Sicht der Numerik interessant. Indem man solche Triangulationen in geeigneter Weise als Graphen betrachtet, spiegeln sie topologische, analytische und kombinatorische Eigenschaften der Fäche wieder. 1972 zeigte Altshuler, dass in gewissen Triangulationen eines Torus stets ein Hamilton-Kreis zu finden ist. Diese Beobachtung zum Ausgangspunkt nehmend, werden in der vorliegenden Arbeit reguläre Graphen untersucht, die als Triangulationen von allgemeinen geschlossenen Flächen auftreten. Ein Hamilton-Kreis einer solchen Triangulierung, der eine topologische Kreisscheibe berandet, heisst kontrahierbar. Es gelingt dem Autor eine notwendige und hinreichende Bedingung für die Existenz eines kontrahierbaren Hamilton-Kreises anzugeben. Dabei spielt ein Baumgraph in der dualen Triangulierung eine entscheidende Rolle.

triangulation or a degree-regular triangulation of type q. Please see [14] for details about graphs on surfaces and [5] for related terminology in graph theory.

In this article, we are interested in studying cycles, especially Hamiltonian cycles, which are in the edge graphs of equivelar triangulations of surfaces. Such cycles have been extensively studied in the plane. For example, in [20], Tutte showed that every 4-connected planar graph has a Hamiltonian cycle. In 1970, Grünbaum conjectured that every 4-connected graph that admits an embedding in the torus has a Hamiltonian cycle, (see [9] and [10]). In particular, this conjecture includes the 4-connected graphs whose embedding gives rise to equivelar maps on the torus. It is well known that there are exactly three distinct types of equivelar maps on the torus, namely, $\{3, 6\}$, $\{4, 4\}$ and $\{6, 3\}$, where the last one is the dual of the $\{3, 6\}$ map, (see [7] and [8]).

A. Altshuler studied Hamiltonian cycles and paths in the edge graphs of equivelar maps on the torus, that is, in the maps that are equivelar and of types $\{3, 6\}$ or $\{4, 4\}$ (see [1], [2]). He showed that, in the graph consisting of vertices and edges of equivelar maps of the above types there exists a Hamiltonian cycle. He also showed that a Hamiltonian cycle exists in every 6-connected graph on the torus.

By definition the faces in a triangulated surface are contractible in the topological sense. A collection H of these faces is called a *contractible* sub-complex of the triangulation if the union of the elements of H is contractible. A cycle in the edge graph of a triangulation will be called contractible if the union of triangles that is bounded by it on one side is a contractible sub-complex, (see [19]). There are other definitions of contractible sub-complexes, see, e.g., [18], p. 744. Nevertheless, we will follow the above definition in this paper.

In [4], Barnette showed that any 3-connected graph other than K_4 or K_5 contains either a contractible cycle or a simple configuration as a subgraph. The definition of contractibility in article [4] asserts that, after contraction, the connectivity of the graph remains unchanged (please see [6], p. 111 for the definition of edge-contraction).

It is also well known that triangulations of surfaces are 3-connected. Because we are motivated by works of Grünbaum, Altshuler and Barnette, we combine these two concepts: Hamiltonicity and the contractibility of a cycle. Furthermore we ask ourselves whether we can always find a contractible Hamiltonian cycle in a given equivelar triangulation of a surface.

In this article, we present a necessary and sufficient condition for the existence of such a cycle in the edge graph of a given equivelar triangulation of surfaces (Theorem 1). We have strong reasons [12] to believe that the result given in this article will lead to an algorithm that can be used to find contractible Hamiltonian cycles in general triangulations and maps of surfaces. In addition, the existence of Hamiltonian cycles and, in particular, contractible Hamiltonian cycles assumes significance in light of the following two examples.

In a triangulation, two triangles with a common edge form a quadrilateral with one diagonal. By replacing the existing diagonal with the other one, one obtains a different triangulation on the same surface. This process is called a *diagonal flip*. In [16] it is shown that, if $n \geq 5$, then any two n vertex triangulations on the sphere that has a Hamiltonian cycle can be transformed into each other by at most $4n - 20$ diagonal flips. In addition it is shown that these flips preserve the existence of Hamilton cycles. Moreover, in [15] the authors used contractible Hamiltonian cycles in triangulations of a projective plane to prove that any two triangulations on a projective plane with n vertices can be transformed into each other by at most $8n - 26$ diagonal flips. The techniques can be further explored to obtain or improve a bound on the number of diagonal flips required to transform an n-vertex triangulation of a fixed surface S into another n-vertex triangulation of S.

Given an equivelar triangulation of a surface S that contains a contractible Hamiltonian cycle, we show that there exists a certain type of tree in the edge graph of the dual map of the given triangulation. Conversely if such a tree exists in the dual map of a triangulation, we show that the given triangulation has a Hamiltonian cycle which bounds a triangulated 2-disk. If the equivelar triangulation of a surface has n vertices, then this disk has exactly $n - 2$ triangles and all of its n vertices lie on the boundary cycle. We begin with some definitions.

2 Definitions and Preliminaries

In this section we present some definitions that will be needed in the course of the proof of Theorem 1. For more details on these definitions, please refer to [13] and [17]. A map is called a *Simplicial Complex* if each of its faces is a simplex. Thus a triangulation is a *Simplicial Complex*. For a simplicial complex K, the graph consisting of its edges and vertices is called the *edge-graph* of K and is denoted by $EG(K)$. If v is a vertex of K, then the number of edges that are incident with v is called the *degree* of v and is denoted by $\deg(v)$. If every vertex of K has same degree q then we define degree of K as degree of v and we denote it by $\deg(K) = q$. In the literature, vertices, edges and faces of K are frequently termed as 0, 1 and 2 faces (or simplices) respectively. If the number of i-simplices of a simplicial complex K is $f_i(K)$ where $0 \leq i \leq 2$, then the number $\chi(K) = f_0(K) - f_1(K) + f_2(K)$ is called the *Euler characteristic of* K. The Euler characteristic of a map is defined similarly.

Let K be a simplicial complex. An edge τ of a 2-face σ in K is said to be a *free* 1-face of σ if τ is not contained in any other 2-face in K. The process of removing a 2-face with a free 1-face in a simplicial complex K is called an *elementary collapse* on K. Applying a sequence of elementary collapses to K results in another simplicial complex K', and K is said to collapse to K'. If K collapses to a point, then we say that K is a collapsible simplicial complex. It is a fact that topologically collapsible simplicial complexes are contractible: see, e.g., [11], p. 32.

If K denotes a map on a surface S, then the *dual map* M, of K is defined to be the map on S that has for its vertices the set of faces of K such that two vertices u_1 and u_2 of M are joined by an edge in M if the corresponding faces in K are adjacent. The well-known maps of type $\{3, 6\}$ and $\{6, 3\}$ on the surface of a torus are mutually dual maps: see Example 1 below.

Example 1. $\{3, 5\}$- and $\{3, 6\}$-equivelar maps are respectively shown on the Icosahedron (Fig. 1) and the flat torus (Fig. 2). The dual map of the $\{3, 6\}$ map on the flat torus is also shown. The dashed lines show a proper tree and the darkened lines constitute a contractible Hamiltonian cycle on these two surfaces.

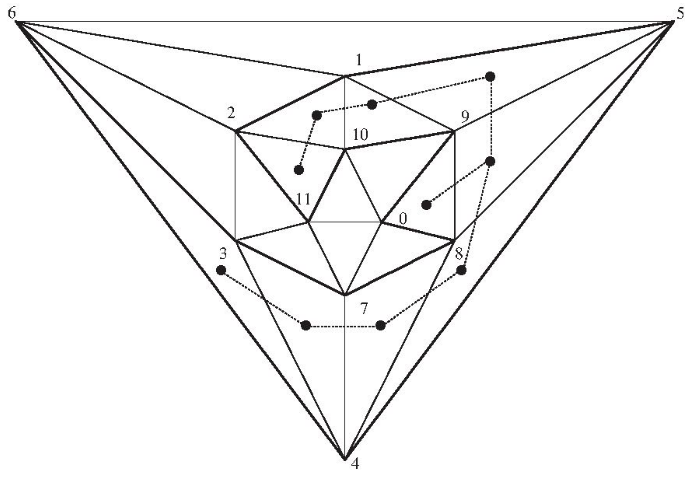

Figure 1

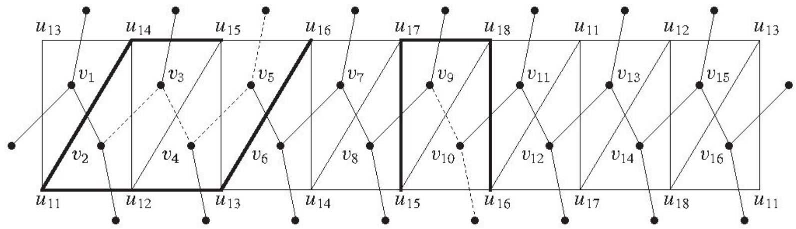

Figure 2

A path P in a graph G is a subgraph $P : [v_1 v_2 \ldots v_n]$ of G, such that the vertex set of P is $V(P) = \{v_1, v_2, \ldots, v_n\} \subseteq V(G)$ and $v_i v_{i+1}$ are edges in P for $1 \leq i \leq n-1$. A path $P : [v_1, v_2, \ldots, v_n]$ in G is said to be a cycle if $v_n v_1$ is also an edge in P. A graph without any cycles or loops is called a tree. The main object of study of this paper is a tree that is defined as follows:

Definition 1. Let M denote a map on a surface S such that M is the dual map of an n-vertex equivelar triangulation K of the surface. Then M is a $\{q, 3\}$-equivelar map for some $q = \deg(K)$. Let T denote a tree on $n-2$ vertices of M (i.e., in the edge graph of M). We say that T is a proper tree if the following two conditions hold:

1. whenever two vertices u_1 and u_2 of T belong to a face F in M, a path $P[u_1 u_2]$ joining u_1 and u_2 in the boundary of F belongs to T;
2. if there exists a path P in T that also lies in a face F of M, then the length of P is at most $q-2$, where $q = \deg(K)$.

In the following section, we present some facts and properties of a proper tree and prove the main result of this article.

3 Results

Let S be a surface with triangulation K and let the corresponding dual map be M. Let $v \in V(T)$ be a vertex in a proper tree T in M. Then, $\deg(v) \leq 3$.

Lemma 3.1. *Let T be a proper tree and let m be the number of vertices of degree three in T. Then, the number of vertices of degree one in T is $m + 2$.*

Proof. We prove this lemma by induction on the number e of edges of T. If $e = 1$, then clearly, the number of vertices of degree one is 2 and there is no vertex of degree 3. Thus, the result is true for $e = 1$. Assume the result to be true for a positive integer $k > 1$. Let T be a proper tree with $e = k + 1$ edges, which are denoted by $e_1, e_2, \ldots, e_k, e_{k+1}$. Let T' be a subtree of T with k edges. Without loss of generality, we may assume that T' consists of the edges $e_1, e_2, \ldots, e_k$ and their vertices. Therefore, the subtree satisfies the induction hypothesis. Because the degree of each vertex in T cannot be greater than 3, the addition of e_{k+1} to T' either results in a new vertex of degree three or a new vertex of degree two. In both of these cases, the statement of the lemma holds. Thus, the proof follows by induction. □

Lemma 3.2. *Let T be an $(n-2)$-vertex proper tree in a polyhedral map M of type $\{q, 3\}$ on a surface S. Then, T intersects every face of M.*

Proof. Let e denote the number of vertices of degree one in T. By the definition of a proper tree, it is clear that M is the dual map of an n-vertex q-equivelar triangulation of S. Hence, T has $n - 2$ vertices and $n - 3$ edges. We claim that the $n - 3$ edges of T lie within exactly $n - e$ faces of M.

To prove this claim, we enumerate the number of faces of M with which the edges of T are incident. We construct sets E and $\tilde{F}$ as follows: let E be a singleton set that contains an edge e_1 of T and let F_1 and F_2 be the adjacent faces of e_1. Let $\tilde{F} := \{F_1, F_2\}$. Add an adjacent edge e_2 of e_1 to E. Then, there is exactly one face F_3 that is distinct from F_1 and F_2 such that e_2 lies in F_3. Add this face to the set $\tilde{F}$ to obtain $\tilde{F} := \{F_1, F_2, F_3\}$. In this way, we successively add edges to the set E that are adjacent to edges in E till we exhaust all of the edges of T. Each additional edge that is added to E contributes exactly one face to the set $\tilde{F}$ unless it is adjacent to two edges in the set E. Thus, the number of faces in $\tilde{F}$ is the number of edges of T minus the number of vertices of degree three $+1$. In a 3-tree, the number of vertices of degree 3 is two less than the number of end points. Therefore, the number of elements in $\tilde{F}$ is $n - 3 - (e - 2) + 1$. That is, $\tilde{F}$ has $n - e$ elements.

Let $F(M)$ denote the set of all faces of M and let $G = F(M) \setminus \tilde{F}$. Then, G has e elements. We claim that an end vertex of T lies on exactly one face F that is contained in G. Observe that each vertex u of T is incident with exactly three distinct faces F_1, F_2 and F_3 of M. The edge of T that is incident with u lies within two of these faces, say, F_1 and F_2: i.e., F_1 and F_2 are in $\tilde{F}$. Because, u is an end vertex, there is no edge of T that is incident with F_3, for otherwise the definition of T would be violated. Thus, u is incident with exactly one face F_3 of M such that F_3 is contained in G. As u is an arbitrary end point, this hypothesis holds for all of the end vertices. If it so happens that, for some end vertices u_1 and u_2 of T, the corresponding equal faces W_1 and W_2 lie in G, then we would have u_1 and u_2 lying on the same face of M, but no path on W_1 that joins u_1 and u_2 lies in T. This occurrence would contradict the definition of T. As a result G has exactly e distinct elements, which proves the lemma. □

Lemma 3.3. *Let K be an n-vertex equivelar triangulation of a surface S. Let M denote the dual map corresponding to K and let T be an $(n-2)$-vertex proper tree in M. Let D denote the sub-complex of K which is dual to T. Then, D is a triangulated 2-disk and the boundary of D, $bd(D)$, is a Hamiltonian cycle in K.*

Proof. By the definition of a dual, D consists of $n-2$ triangles that correspond to the $n-2$ vertices of T. Two triangles in D have an edge in common if the corresponding vertices are adjacent in T. It is easy to see that D is a collapsible simplicial complex, and hence, it is a triangulated 2-disk.

Moreover, as T has vertices of degree one, $bd(D)$ is non-empty. As $bd(D)$ is boundary complex of a 2-disk it is a connected cycle. Observe that the number of edges in $n-2$ triangles is $3(n-2)$ and that for each edge of T, exactly 2 edges are identified. Hence, the number of edges that remain unidentified (i.e., the number of free edges) in D is $3(n-2)-2(n-3)=n$. These edges are precisely those that belong to $bd(D)$. A similar argument shows that the number of vertices in $bd(D)$ is also n. Now, we want to show that all of the n vertices are distinct. For this purpose, assume that there are vertices v_1 and v_2 in $bd(D)$ such that $v_1 = v_2$ and v_1 and v_2 lie on a path of positive length $< n$ in $bd(D)$. This assumption would imply that there are faces F_1 and F_2 in D such that v_1 is in F_1, v_2 is in F_2, F_1 is distinct from F_2 and F_1 is not adjacent to F_2. Thus, there exists a face F' in D such that the vertex $u_{F'}$ in T that corresponds to F' does not belong to the face $F(v_1)$ that corresponds to vertex v_1. However, this conclusion contradicts the fact that T is a proper tree. Therefore, the cycle $bd(D)$ contains exactly n distinct vertices. As the number of vertices $V(K)$ in K is n, $bd(D)$ is a Hamiltonian cycle in K. □

Theorem 3.4. *Let S denote a surface that has an equivelar triangulation K. The edge graph $EG(K)$ of K has a contractible Hamiltonian cycle if and only if the edge graph of the corresponding dual map M of K has a proper tree.*

Proof. The above lemma, Lemma 3.3 proves one half of this theorem. To prove the other half, let K denote an equivelar triangulation and let $H := (v_1, v_2, v_3, \ldots, v_n)$ denote a contractible Hamiltonian cycle in $EG(K)$. Let $\tau_1, \tau_2, \ldots, \tau_m$ denote the faces of a triangulated disk D whose boundary is H. We claim that all of the triangles that triangulate the disc have their vertices on the boundary of the disk, i.e., on H. To prove the claim assume otherwise. Then, there will be identifications on the surface, as all of the vertices of K will also be on H. Now, if x denotes the number of triangles in this disk, then the Euler characteristic relation gives us $1 = n - [\frac{(3\times x)-n}{2} + n] + x$. Thus, $x = n-2$. As a result the number of triangles that triangulate the disc $m = n-2$. Now, in the edge graph of the dual map M of K, consider the graph corresponding to this disk whose vertices correspond to the faces $\tau_1, \tau_2, \ldots, \tau_m$. It is easy to check that this graph is a tree that is also a proper tree. □

Remark. Because we started with the results of Altshuler, in the present article, we have confined ourselves to degree-regular triangulations. In [12], we have been able to show that the result also holds for triangulations of surfaces where the degree of each vertex is at least 4. Moreover, for some class of maps on surfaces, we have been able to show a similar result with a slight modification in the definition of a proper tree.

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References

[1] A. Altshuler, Construction and enumeration of regular maps on the torus, *Discrete Math.* (**4**) (1973), 201–217.

[2] A. Altshuler, Hamiltonian circuits in some maps on the torus, *Discrete Math.* (**4**) vol. 1, (1972), 299–314.

[3] D. Barnette, Personal Communications.

[4] D. Barnette, Contractible circuits in 3-connected graphs, *Discrete Math.***187**, (1998), 19–29.

[5] J.A. Bondy and U.S.R. Murthy, *Graph theory with applications*, North Holland, Amsterdam, 1982.

[6] R.B. Borie, R.G. Parker and C.A. Tovey, Recursively constructed graphs, in *Handbook of Graph Theory*, J.L. Gross and J. Yellen (eds.), CRC Press, (2004), 99–118.

[7] U. Brehm and W. Kuhnel, Equivelar maps on the torus, *European J. Comb.*, **29**, (2008), 1843–1861.

[8] B. Datta and N. Nilakantan, Equivelar polyhedra with few vertices, *Discrete & Comput Geom.* **26**, (2001), 429–461.

[9] R.A. Duke, On the Genus and Connectivity of Hamiltonian Graphs, *Discrete Math.*, **2**, (1972), 199–206.

[10] B. Grünbaum, Polytopes, Graphs and Complexes, *Bull. Am. Math. Soc.*, **76**, (1970), 1131–1201.

[11] M. Hachimori, *Combinatorics of Constructible Complexes*, Doctoral Thesis, Univ. of Tokyo, (2000).

[12] D. Maity and A.K. Upadhyay, *Contractible Hamiltonian cycles in maps on surfaces*, (in preparation).

[13] P. McMullen and E. Schulte, *Abstract Regular Polytopes*, CUP, 2002.

[14] B. Mohar and C. Thomassen, *Graphs on Surfaces*, The John Hopkins Univ. Press, 2001.

[15] R. Mori and A. Nakamoto, Diagonal flips in Hamiltonian triangulations on the projective plane, *Discrete Math.*, **303**, (2005), 142–153.

[16] R. Mori, A. Nakamoto and K. Ota Diagonal flips in Hamiltonian triangulations on the sphere, *Graphs & Comb.*, **19**, (2003), 413–418.

[17] J.R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, California, 1984.

[18] S. Negami, Triangulations, in *Handbook of Graph Theory*, J.L. Gross and J. Yellen (eds.), CRC Press, (2004), 737–760.

[19] T. Pisanski and P. Potočnik, Graphs on surfaces, in *Handbook of Graph Theory*, J. L. Gross and J. Yellen (eds.), CRC Press, (2004), 611–624.

[20] W.T. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.*, **82**, (1956), 99–116.

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