The Gergonne and Soddy lines

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The Gergonne and Soddy lines

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1 Introduction

1.1. Given 3 pairwise externally tangent circles $\mathcal{A}, \mathcal{B}, \mathcal{C}$, centered at A, B, C, there are two other circles, called the Soddy circles, which are tangent to these three. The kissing points of the circles T, U, V are then the points of tangency of the incircle $\mathcal{I}$ to the sides of $\triangle ABC$.

One way to make the construction of the Soddy circles is as follows. First, invert $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in another circle $\mathcal{K}$, centered at one of the points of tangency. Now, the original configuration of 3 circles is transformed to two parallel lines and a circle tangent to both (see Figure 1). It is now clear that we can surround this circle with two equal radii circles tangent to this one and the two lines. Finally, invert these two new circles in $\mathcal{K}$ to obtain the Soddy circles $\mathcal{S}, \mathcal{S}'$.

The arrangement of the Soddy circles with respect to $\mathcal{A}, \mathcal{B}, \mathcal{C}$ has been captured in the first stanza of Soddy's *The Kiss Precise*

Neben der bekannten Euler-Geraden hält das Dreieck weitere interessante Geraden bereit. Auf der Soddy-Geraden liegen z.B. die Eppstein-Punkte, der Gergonne-Punkt und sie schneidet die Euler-Gerade im de Longchamps-Punkt. Die Gergonne-Gerade schneidet die Soddy-Gerade im Fletscher-Punkt senkrecht und hat mit der Euler-Geraden den Evans-Punkt gemeinsam. Die Gergonne-Gerade ist die perspektive Achse des Dreiecks und seines Inkreisdreiecks. Der Autor der vorliegenden Arbeit bringt die Soddy- und die Gergonne-Gerade miteinander in Verbindung, indem er sie als geometrischen Ort der Zentren von Kreisen zweier orthogonaler Büschel betrachtet. Eines dieser Büschel enthält die beiden Soddy-Kreise: Diese berühren drei paarweise tangentiale Kreise, die in den Ecken des Dreiecks zentriert sind.

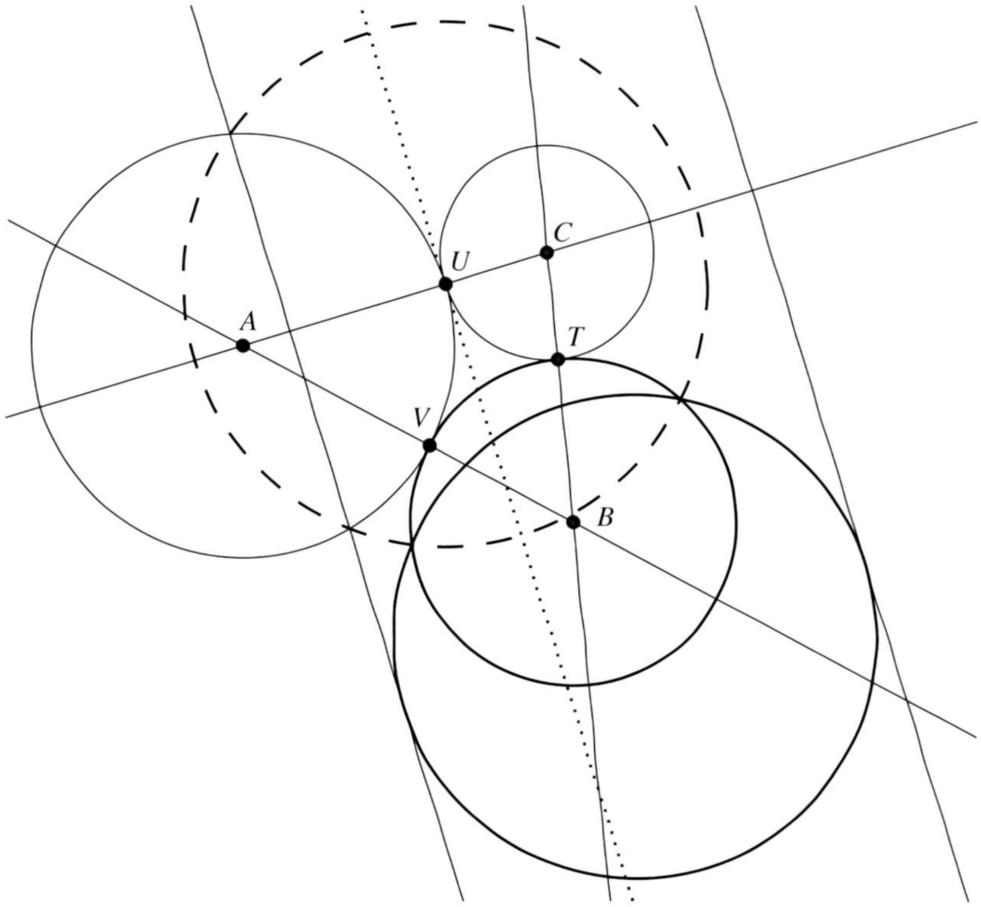

Fig. 1 Tangent circles with centers A, B, C and their inversions in a circle centered at U

For pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.

Nobel prize winner Frederick Soddy [4] investigated and popularized these circles. However, as noted by the referee, the Soddy circles actually have been known since antiquity through the work of Apollonius and then later Pappus; constructions of these circles by ruler and compass have been shown by Viete (1600) and then Gergonne (1816).

Note that if the mutually tangent circles $\mathcal{A}, \mathcal{B}, \mathcal{C}$ do not have disjoint interiors then we may use an excenter and excircle of the triangle $\triangle ABC$ to obtain the Soddy circles tangent to these three.

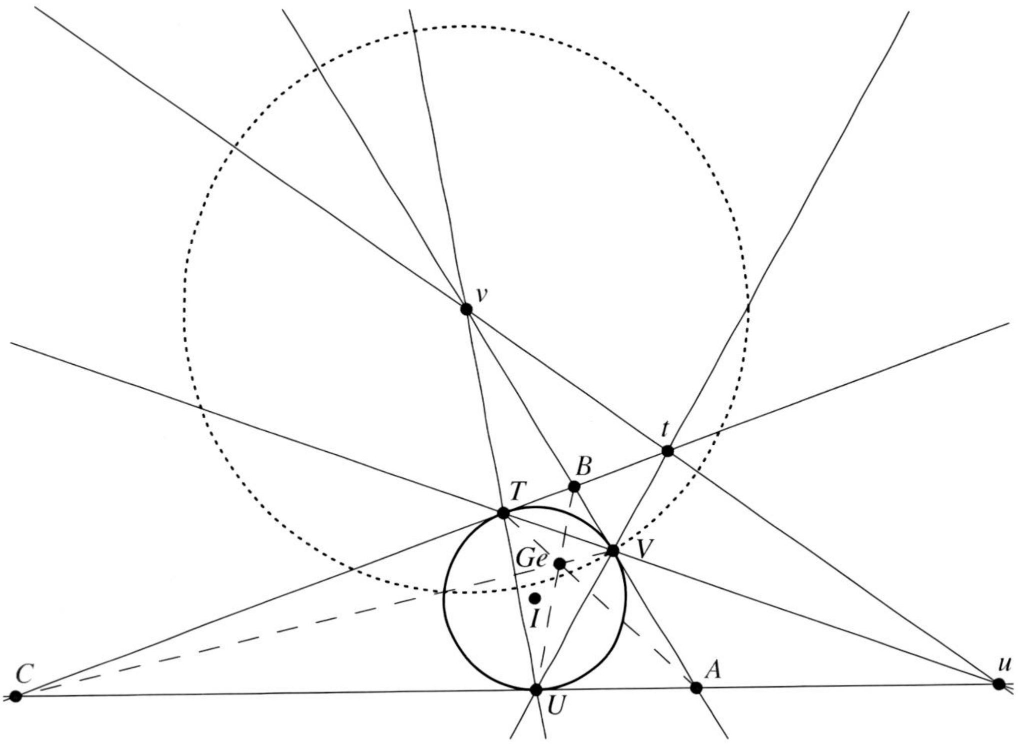

Fig. 2 Gergonne line

1.2 The Gergonne Line

The triangles $\triangle ABC$, $\triangle TUV$ are in perspective from the Gergonne point Ge, and hence by Desargues' Theorem the triangles are in perspective from a line, called the Gergonne line $\mathcal{G}$.

By perspectivity the lines TU and AB meet at a point of $\mathcal{G}$, denoted by v. In a similar way we determine t, u. See Figure 2.

1.3 The Soddy Line

The line passing through the centers S, S' of the Soddy circles $\mathcal{S}$, $\mathcal{S}'$ is the Soddy line. The Soddy line also passes through the incenter I and the Gergonne point Ge of $\triangle ABC$; the points I and Ge are harmonic conjugates with respect to S, S' [2].

2 A Pencil of Circles

Suppose we are given a circle $\mathcal{K}$, centered at K and an interior point $P \neq K$. The polar dual of P is a line, say L. For any point Q on L its polar line intersects $\mathcal{K}$ at the tangencies Z, Z' from Q. Consider the circle C_Q with center Q and passing through Z, Z'. The set of these circles for varying Q on L form a pencil $\mathcal{K}_P$. (An alternative way to construct the elements of $\mathcal{K}_P$: for any point $Z \in K$ the tangent line there meets the line at Q and the circle with center Q passing through Z belongs to the pencil $\mathcal{K}_P$.)

The radical of the pencil is the line perpendicular to L and passing through P and K. (For external P the radical line has no real intersections with the pencil.)

2.1 We shall use the properties of a polarity with respect to a conic [3]. The polarity with respect to a conic $\mathcal{K}$ induces the symmetric notion of conjugacy: P is conjugate to Q iff P lies on the polar of Q. A line m meets the conic $\mathcal{K}$ in two points M, M'. Then $A \in m$ is conjugate to $A' \in m$ iff the harmonic conjugate of A is A', i.e., the cross ratio $(A, A'; M, M') = -1$.

Theorem 2.1. *Any circle in the pencil $\mathcal{K}_P$ is orthogonal to $\mathcal{K}$.*
The radical line of the pencil is KP.
The polar of P in any circle of $\mathcal{K}_P$ passes through K. That is, P and K are conjugate with respect to any circle of the pencil $\mathcal{K}_P$.

Proof. Since the center of a circle in the pencil $\mathcal{K}_P$ lies on a tangent to $\mathcal{K}$ (by construction), the circle is orthogonal to $\mathcal{K}$.
Since $\mathcal{K}$ and a circle in the pencil with center are orthogonal then the power of K is the radius of $\mathcal{K}$. The point P lies on a common secant of $\mathcal{K}$ and a circle $\mathcal{M}$ of $\mathcal{K}_P$, thus the power of P with respect to $\mathcal{K}$ is the same as the power of P with respect to $\mathcal{M}$. Thus the power of P, K is constant for the circles of $\mathcal{K}_P$ so both of these points lie on the radical, which is perpendicular to the line of centers.
The point P lies on a common secant with $\mathcal{K}$ and a circle of the pencil $\mathcal{K}_P$; so K is on the polar of P since the circles are orthogonal. Hence K and P are conjugate. So the result follows by symmetry of the conjugacy relation. □

The following is immediate from the theorem and the preceding remarks.

Corollary 2.2. *The radical axis of $\mathcal{K}_P$ meets any circle of the pencil in points R, R' which are harmonic conjugates with respect to K and P.*

The referee has pointed out that this theorem and corollary have been known since Steiner [5], see especially, Fig. 15, p. 185, and pp. 168–9.

2.2 Gergonne Pencil

Consider the triangle $\triangle ABC$ and its incircle $\mathcal{I}$. The pencil of circles $\mathcal{I}_{Ge}$ is called the Gergonne pencil.

Proposition 2.3. *The polar lines of the points t, u, v are respectively AT, BU, CV with respect to the incircle $\mathcal{I}$. Thus the polar of Ge is the line $\mathcal{G}$.*

Proof. We prove this for the point v; the others are shown similarly. By construction, the line ABV is tangent to the incircle at V and passes through v, so V is on the polar of v. The polar of C is TU, which passes through v. Thus C and v are conjugate; hence the polar of v passes through C. Thus the polar of v is CV.
The lines AT, BU, CV meet at the Gergonne point and hence the polar of Ge is the line tuv, the Gergonne line $\mathcal{G}$. □

Theorem 2.4. *The Gergonne pencil is a pencil of circles orthogonal to $\mathcal{I}$.*
The line of centers of the Gergonne pencil is the Gergonne line.
The radical line of the Gergonne pencil is the Soddy line

Proof. The first statement follows immediately from Theorem 2.1
For the second part, by definition, the line of centers of the pencil $\mathcal{I}_{Ge}$ is the polar of Ge with respect to the incircle $\mathcal{I}$. By the previous proposition, this polar has been identified as the Gergonne line.
By Theorem 2.1 the radical line is IGe. The Soddy line also passes through I and Ge, [2]. □

The next result follows from the theorem above and the previous corollary.

Corollary 2.5. *The common points R, R' of the Gergonne pencil are harmonic conjugates with respect to I, Ge.*

The intersection of the Soddy line and Gergonne line is called the Fletcher point, Fl [2]; it is the midpoint of R and R'. It follows from these results that Fl is simply the inversion of Ge in the incircle $\mathcal{I}$ [3].

The referee has suggested that one may obtain a synthetic proof of the results of [2] used in the proof of the theorem above, by using our results and those of Steiner, [6] p. 157.

2.3 Soddy Construction

We construct circles $\mathcal{A}', \mathcal{B}', \mathcal{C}'$ which are orthogonal to $\mathcal{I}$, centered at t, u, v respectively and passing through T, U, V respectively. These circles are in the Gergonne pencil (Figure 2) by definition of the pencil, since the tangent to $\mathcal{I}$ at T passes through t. We may now use these circles to construct the Soddy circles as described next.

The Soddy circles $\mathcal{S}, \mathcal{S}'$ are tangent to each $\mathcal{A}, \mathcal{B}, \mathcal{C}$ at points a, b, c, and a', b', c' respectively. We construct the Soddy circles by constructing these points of tangency with $\mathcal{A}, \mathcal{B}, \mathcal{C}$ using the inversion process. Construct a circle of inversion centered at U as in Figure 1. The inversion of $\mathcal{A}, \mathcal{C}$ are parallel lines $\mathcal{A}^*, \mathcal{C}^*$, and the inversion of $\mathcal{B}$ is a circle $\mathcal{B}^*$ tangent to these lines. Since angles, incidences and tangencies are preserved by inversion, the transform of $\mathcal{I}$ is the line $\mathcal{I}^*$ perpendicular to $\mathcal{A}^*, \mathcal{C}^*$ which is also a diameter of $\mathcal{B}^*$, meeting these parallel lines at the transforms T', V', of T, V respectively (recall that the transform X' of a point X lies on a line through U). Now construct the circle $\mathcal{A}^0$ centered at T' passing through V' (see Figure 3). This circle meets the line $\mathcal{A}^*$ at two points which transform back to a, a' on $\mathcal{A}$, giving the points of incidence with the Soddy circles as described in 1.1. The transform of the circle $\mathcal{A}^0$ is orthogonal to $\mathcal{I}$ and passes through T having its center on UV, hence its center is t; thus $\mathcal{A}^0 = \mathcal{A}'$.

Thus we obtain the tangencies a, a', b, b', c, c' as the intersections of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ with $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$ as described above and hence the Soddy circles are $\mathcal{S} = abc$, $\mathcal{S}' = a'b'c'$. It is now easy to see that the Soddy circles are ruler-compass constructible.

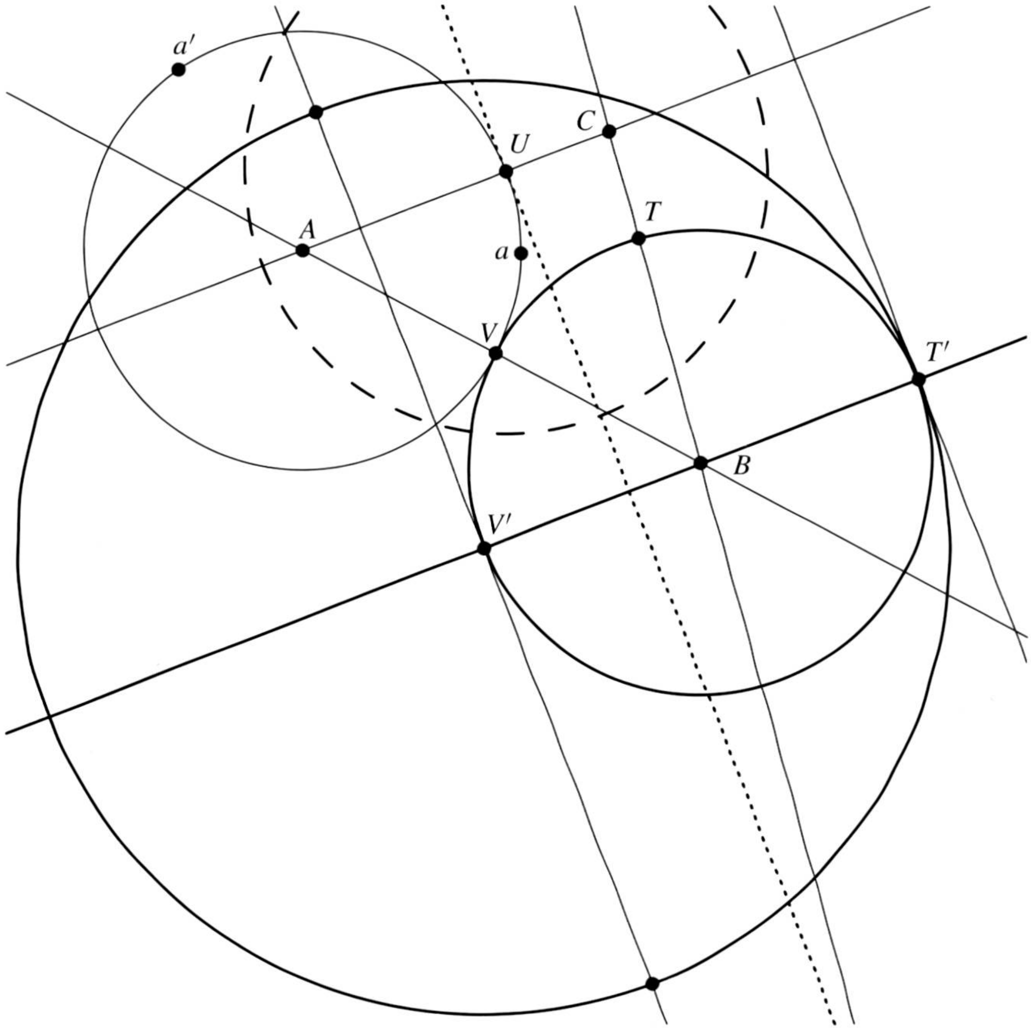

Fig. 3 Inversion constructions

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