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Tiling by hyperbolic dominoes

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1 Introduction

A quite popular tiling problem is that of the classic chessboard with two diagonally opposite corners removed: can this "truncated chessboard" be covered by dominoes? Every domino should cover exactly two (vertically or horizontally) adjacent squares. A simple argument for proving that such a tiling is not possible is by noticing that a domino of any orientation covers one black square and one white square, whereas the "truncated chessboard" has 32 squares of one color and 30 of the other color.

For which kind of boards does a tiling by dominoes exist? As in the previous example, the board should be a union of squares in such a way that we can color them alternately white and black, and there are exactly as many white squares as black squares. This condition is necessary, but it is not sufficient (see Figure 1).

[1]Research supported by CIC-UMSNH

Bekanntlich lässt sich ein 8×8 Schachbrett, bei dem zwei diagonal gegenüberliegende Eckfelder entfernt wurden, nicht lückenlos mit 2×1 Dominosteinen bedecken. Eine notwendige Bedingung für die Existenz einer solchen Parkettierung bei einem zusammenhängenden Schachbrett beliebiger Form ist, dass es gleichviele schwarze wie weisse Felder enthält, denn jeder Dominostein bedeckt ja immer zwei Felder unterschiedlicher Farbe. Diese Bedingung ist jedoch nicht hinreichend. Thurston hat 1990 ein entsprechendes allgemeines Kriterium angegeben. Die Autoren der vorliegenden Arbeit zeigen Thurstons Methode in neuem Licht und verallgemeinern sie auf die reizvollen Gitter in der hyperbolischen Ebene.

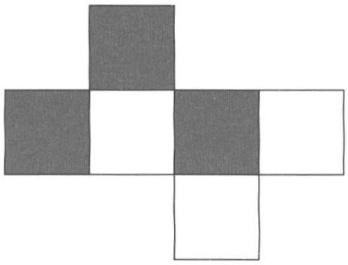

Figure 1: A board that does not admit a tiling by dominoes.

The aim of this article is to give a purely combinatorial proof of a criterion which allows to decide whether a given domain may be covered by dominoes; this is a criterion given by Thurston. On the other hand, our proof applies not only to the Euclidean case but also to the spherical and hyperbolic geometries.

Thurston [7] developed the so-called *height functions* as a major tool to prove his criterion in the Euclidean case. Roughly speaking, a polygon admits such a tiling if and only if the height function over the polygon boundary is a Lipschitz function with Lipschitz constant $K = 1$ (see inequality (1)). In fact, Thurston was able to find an algorithm to produce a tiling when it exists, and the algorithm also indicates the nontileability when a tiling does not exist. Section 2 is devoted to reviewing the concept of height function, Thurston's results, and their adaptation to spherical and hyperbolic cases.

Thurston's algorithm is explained in Section 3.

Two examples are reviewed in Section 4.

Height functions arose when Thurston regarded certain Cayley graphs as graphs in $\mathbb{R}^3$. Our combinatorial approach makes it clear that the original algebraic framework using Cayley graphs is not entirely necessary. This can also be seen in [5]. However, for the sake of completeness, in Section 5 we briefly summarize the connection between height functions and Cayley graphs of Conway tiling groups in the Euclidean case.

We refer to [1] for the hyperbolic geometry background. In our figures, we shall use the Poincaré disk model $\mathbb{D}$ for the hyperbolic plane [1, Section 2.7]. We expect that a reader who has not studied hyperbolic geometry ought to be able to go through this note recalling two facts: *hyperbolic lines* in $\mathbb{D}$ are arcs of Euclidean circles orthogonal to the boundary $\partial\mathbb{D}$ (including straight lines passing through the origin), and *angles between hyperbolic lines* are equal to Euclidean angles between circles.

1.1 The concrete problem

In the Euclidean plane there are solely three regular grids up to similarity: the triangular, the square and the hexagonal grids (see Figure 2). In that regard, the hyperbolic plane is much richer than the Euclidean one. Recall the *hyperbolic grid* $\{p, q\}$, where p and q are positive integers with $p, q \geq 3$, to be a tiling of the hyperbolic plane by a hyperbolic regular p-gon with angle $2\pi/q$ (see Figure 3). It is understood that the intersection of two p-gons is either a complete edge, or a vertex, or the empty set. The hyperbolic regular p-gon with angle $2\pi/q$ exists if and only if $(p-2)(q-2) > 4$, and it is unique up

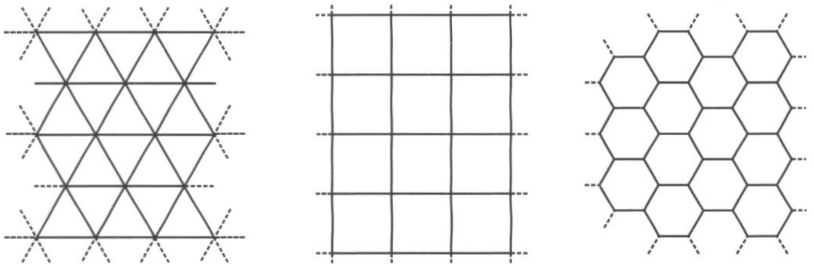

Figure 2: The unique three grids of the Euclidean plane.

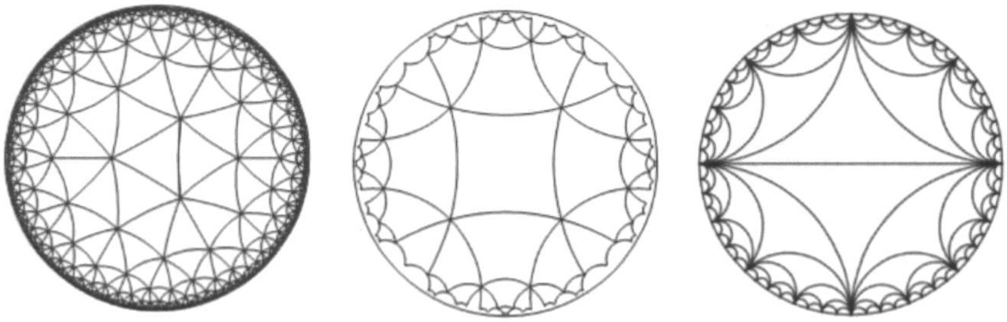

Figure 3: Hyperbolic grids $\{3, 7\}$, $\{4, 6\}$ and $\{3, \infty\}$ in the disk model.

to hyperbolic isometries, in which case the hyperbolic grid $\{p, q\}$ is well defined up to hyperbolic isometries [1, Chapter 6]. We also include ideal p-gons allowing $q = \infty$. When $(p-2)(q-2) = 4$ we have a *Euclidean grid*, and with $(p-2)(q-2) < 4$ results a *spherical grid*, obtained by central projection of a platonic solid over its circumscribed sphere.

In this article we assume that either q is an even integer or $q = \infty$, so we can color p-gons alternately white and black, that is, two adjacent p-gons (that share an edge) have opposite colors, like a chessboard.

By $\{p, q\}$*-domino* we mean the union of two adjacent p-gons in the grid $\{p, q\}$; this is unique up to hyperbolic isometries for $(p-2)(q-2) > 4$. Hence a $\{4, 4\}$-domino is the usual (Euclidean) domino, and a $\{3, 6\}$-domino is the so-called lozenge.

Given a grid $\{p, q\}$, a *grid-path* is a differentiable (of class C^1) path $\gamma : [a, b] \to \mathbb{D}$ which satisfies two conditions: the image is union of complete edges of the grid, and $\gamma'(t) = 0$ for some $t \in [a, b]$ implies that $\gamma(t)$ is a vertex of the grid. The second condition ensures that the path runs along the complete edges before stopping, hence it determines an orientation of the edges while passing through them; although some edges may be traversed more than once in different directions. Then a grid-path is composed of *edge-paths* in an obvious way. A *closed grid-path* is defined as a grid-path starting and ending at the same vertex of the grid. We say that a closed grid-path is *simple* if it has no self-intersections.

A closed subset B of the hyperbolic plane is a $\{p, q\}$*-board* (or simply a *board*) if its boundary is a simple closed grid-path on the grid $\{p, q\}$.

Thurston's criterion responds to the question: *when a* $\{p, q\}$*-board can be tiled by* $\{p, q\}$*-dominoes?* (See Figure 4.) Compare [3], where essentially the same question is addressed with a somewhat different approach.

Figure 4: *Left*: A tiling by $\{5,4\}$-dominoes. Each dotted line represents the midline of one domino. *Right:* A perfect matching or dimer covering determined by that tiling.

The below arguments apply equally to Euclidean, spherical and hyperbolic geometries. Only the grids $\{4,4\}$, $\{3,6\}$ and $\{6,3\}$ are Euclidean; $\{3,3\}$, $\{4,3\}$, $\{3,4\}$, $\{5,3\}$ and $\{3,5\}$ are spherical; and all others are hyperbolic. Since we assumed that q is even, in order to have bicolored boards, we leave out grids $\{6,3\}$, $\{3,3\}$, $\{4,3\}$, $\{5,3\}$ and $\{3,5\}$. Therefore the spherical case is not interesting because every board with the same number of white and black triangles in the octahedral grid can be tiled by dominoes. We invite the reader to check it.

1.2 Connection with graphs and physics

Tileability of a board by dominoes is a geometric realization of a classical combinatorial concept: *perfect matching* of a graph. We turn a board into a graph by replacing the p-gons by vertices and putting an edge between those vertices which correspond to adjacent p-gons; the graph G obtained is the *dual graph* of the board. Then, tiling the board by dominoes corresponds to selecting edges from G such that every vertex is the endpoint of exactly one of the chosen edges (see Figure 4). Such selection of edges is known as a perfect matching.

Another interesting aspect of dominoes has to do with *dimers*. A dimer is a polymer with two atoms. One may regard each vertex of G as an atom, and each edge in a perfect matching as a representation of a diatomic molecule; so a perfect matching is also known as a *dimer covering*. Height functions have applications to physics; to name one, they can be used to sample randomly a dimer covering with the uniform distribution (see [4] for instance). Perhaps adapting to the hyperbolic geometry can motivate new applications.

Although it is possible to define height functions for dimers on any bipartite graph (see [6] for example), the generalization stated here is almost straightforward, is not necessary to develop additional concepts.

1.3 Acknowledgement

We thank the referee for comments and suggestions that really helped to improve the article. We also thank Scott Vorthmann for providing our personalized license for vZome 4.0. This software was used to create Figures 7, 8 and 9.

2 Thurston's criterion

Let B be a bicolored $\{p, q\}$-board. Let V denote the set of vertices of the grid which lie at B. We consider two vertices in V to be *adjacent* only if they are connected by an edge which is contained in B. We denote by $[u, v]$ an edge-path from u to v, where u and v are adjacent vertices. An edge-path is positively oriented if it has a black p-gon on its left.

Definition 1. A *height function* h on B is a function $h : V \to \mathbb{Z}$ which satisfies:

1. if $u, v \in V$ are adjacent vertices and the edge $[u, v]$ is positively oriented, then $h(v) = h(u) + 1$ or $h(v) = h(u) + 1 - p$;
2. if in addition $[u, v]$ is part of the boundary of R, then just $h(v) = h(u) + 1$.

The (central) Figure 5 illustrates a height function.

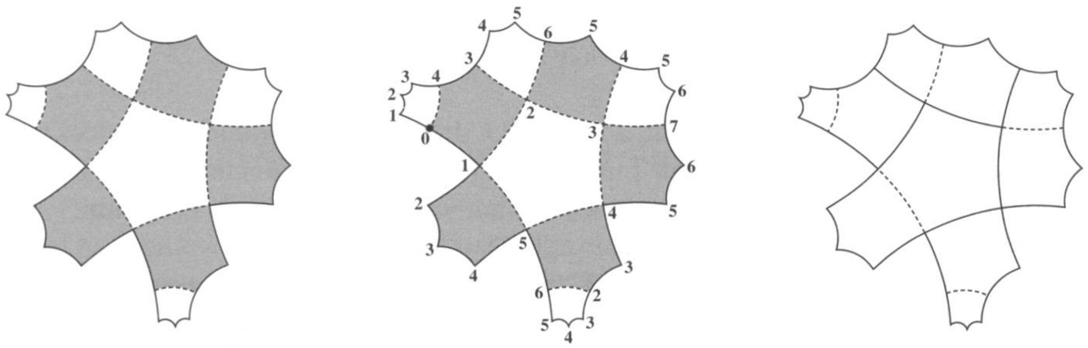

Figure 5: *Left:* A bicolored {5, 4}-board. *Center:* The values of a height function H defined on the board. *Right:* Edges whose ends have difference 4 are dotted lines, and the other edges are delineated lines.

Proposition 2. *Let B be a bicolored $\{p, q\}$-board. If there exists a height function defined on B, then B can be tiled by $\{p, q\}$-dominoes.*

Proof. Let $h : V \to \mathbb{Z}$ be a height function. Consider an arbitrary p-gon P contained in B. Let $v_1, v_2, \ldots, v_p$ denote the vertices of P, labeled so that v_{i-1} and v_i are adjacent and $[v_{i-1}, v_i]$ is positively oriented, for any i.

Notice that $h(v_i) = h(v_{i-1}) + 1 - b_i p$, where $b_i \in \{0, 1\}$. Then

$$
\begin{aligned}
h(v_2) &= h(v_1) + 1 - b_2 p, \\
h(v_3) &= h(v_2) + 1 - b_3 p = h(v_1) + 2 - (b_2 + b_3)p, \\
&\vdots \\
h(v_p) &= h(v_1) + (p-1) - (b_2 + \cdots + b_p)p = h(v_1) + p - 1 - np,
\end{aligned}
$$

where $n = b_2 + \cdots + b_p$ is the number of edges such that its ends have difference > 1. It follows $h(v_1) = h(v_p) + 1 - (1 - n)p$, therefore $n = 0$ or $n = 1$. There is, in both cases, exactly one index k such that $h(v_k) = h(v_{k-1}) + 1 - p$, that is $h(v_l) = h(v_{l-1}) + 1$ for any $l \neq k$.

It is now immediate that we obtain a tiling of B by $\{p, q\}$-dominoes by erasing all edges whose ends have difference > 1 (see Figure 5). □

Let B be a bicolored board. Given two vertices u and v in B, a grid-path from u to v is called *positively oriented* if it moves in positive direction through all edge-paths composing it. We also define the *distance* between u and v, denoted by $d(u, v)$, to be the minimal length of all positively oriented grid-paths from u to v contained in B, where the *length* of a grid-path is equal to the number of edges composing it. The distance d is not symmetric, it behaves as distance to go by car from one place u to another v in a city: the senses of the streets prevent $d(u, v) = d(v, u)$. There is always a positively oriented grid-path from u to v, for the same reason that we can get from one place to another by car through the streets of a city: given any path from u to v, we replace each edge-path e traversed in nonpositive sense by a positively oriented grid-path which turns around a p-gon adjacent to e.

Proposition 3. *Let B be a bicolored board and h be a height function defined on B. Then any pair of vertices u, v on the boundary of B satisfies*

$$h(v) - h(u) \leq d(u, v). \tag{1}$$

Later we shall prove a converse result: a function h defined on the boundary vertices which satisfies inequality (1) and condition 2 in the definition of height function can be extended to a height function on B. This equivalence is what we call **Thurston's criterion**.

Proof. Consider the tiling of B given by Proposition 2. The proof is by induction over the number of dominoes needed to tile B.

Suppose that B is a domino. Let u and v be vertices in the boundary of B such that $h(v) > h(u)$. If u and v belong to the same p-gon, then $h(v) - h(u) = d(u, v)$. If u and v lie in different p-gons, we have $d(u', v) \leq d(u, v)$, where u' is the unique vertex on the same p-gon as v such that $h(u) = h(u')$. Thus $h(v) - h(u) = h(v) - h(u') = d(u', v) \leq d(u, v)$.

Now we suppose that inequality (1) is valid for every board which can be tiled by $n - 1$ dominoes, with $n > 1$. Let B be a board tiled by n dominoes. We cut one domino D out of B along a grid-path w_1, in such a way that w_1 divides B into two regions: D and a board B' tiled by $n - 1$ dominoes. Let u and v be vertices on the boundary of B lying in different regions, namely $u \in B'$ and $v \in D$. Consider a grid-path w_2 in B such that $d(u, v)$ is equal to the length of w_2. Let $u_0 \in w_1 \cap w_2$ be another vertex. Then $h(v) - h(u_0) \leq d(u_0, v)$ and $h(u_0) - h(u) \leq d(u, u_0)$ by induction hypothesis. Thus $h(v) - h(u) \leq d(u, u_0) + d(u_0, v)$, but $d(u, v) = d(u, u_0) + d(u_0, v)$ since w_2 is minimal. □

Let us consider a bicolored grid. Let π be a grid-path. We define the *oriented length* of π, denoted by $\ell(\pi)$, to be the number of positively oriented edge-paths composing π minus the number of nonpositively oriented edge-paths composing π.

Proposition 4. *Let u and v be two vertices of the $\{p, q\}$-grid, and π_1 and π_2 be two grid-paths from u to v such that the closed path $\pi_1 \pi_2^{-1}$ is the boundary of a board B. Then $\ell(\pi_1) \equiv \ell(\pi_2) \pmod{p}$.*

Recall the standard notation: π^{-1} is the path π traveled in the reverse direction, and $\pi\tau$ is the concatenation of π and τ, where it is required that the starting point of τ coincides with the ending point of π.

Proof. The proof is by induction on the number of p-gons making up B.

If B is a p-gon by itself, then $\ell(\pi_1) - \ell(\pi_2) = \ell(\pi_1) + \ell(\pi_2^{-1}) \equiv 0 \pmod{p}$.

Now we suppose that the result is valid for boards which are made of $(n-1)$ p-gons. Let B be a board which is made of n p-gons. We cut one p-gon out of B along a grid-path w_0 from x_1 to x_2, where x_1 and x_2 are vertices at π_1. Let us call w_1 and w_2 the subpaths of π_1 that respectively go from u to x_1 and from x_2 to v. Notice that the paths $(w_1 w_0 w_2)\pi_2^{-1}$ and $\pi_1(w_1 w_0 w_2)^{-1}$ respectively enclose $n-1$ and one p-gons, therefore

$$\ell(\pi_1) - \ell(\pi_2) = [\ell(\pi_1) - \ell(w_1 w_0 w_2)] + [\ell(w_1 w_0 w_2) - \ell(\pi_2)] \equiv 0 \pmod{p}. \quad \square$$

Let π be a grid-path. By a *lifting of* π we mean the multivalued function H defined on the vertices of π which is described as follows: if $v_1, \dots, v_n$ denote the consecutive vertices of π so that $\pi = \cup_{i=1}^{n-1}[v_i, v_{i+1}]$, then

$$H(v_k) = \begin{cases} 0 & \text{if } k = 1; \\ \ell(\cup_{i=1}^{k-1}[v_i, v_{i+1}]) & \text{if } k = 2, \dots, n, \end{cases}$$

where $\cup_{i=1}^{k-1}[v_i, v_{i+1}]$ denotes the subpath of π starting at v_1 and ending at v_k. Notice that H is not strictly a function, because there may be self-intersections of π; if the path passes through a vertex more than once, the value of H at that vertex is not necessarily unique. The next result responds when H is in fact a function on the boundary vertices of a board.

Proposition 5. *Let B be a bicolored board, and π be a simple closed grid-path which represents the boundary of B starting and ending at some vertex u. The value of the lifting of π at the final vertex of π is equal to zero if and only if the number of black p-gons contained in B is equal to the number of white p-gons contained in B.*

Proof. Suppose that B encloses the same number of white and black p-gons. Let U be the set of edges that form the part of the black p-gons contained in B, and W the set of edges that form the part of the white p-gons contained in B. Since B contains the same number of white and black p-gons, we have $|U| = |W|$. Moreover, each interior edge belongs to exactly one black p-gon and one white p-gon, therefore π travels by the same number of positively oriented edges than of nonpositively oriented edges. It follows that the value of the lifting of π at the final vertex of π is equal to zero.

The converse is proved following the same idea. $\square$

Theorem 6 (Thurston)**.** *Let B be a bicolored $\{p, q\}$-board. We select arbitrarily one boundary vertex to be the starting and ending point of a simple closed grid-path π which describes the boundary of B. Consider the lifting H of π, and suppose that the value of H at the ending point of π is equal to zero. If $H(v) - H(u) \leq d(u, v)$ holds for any pair of vertices u, v in π, then there is a tiling of B by $\{p, q\}$-dominoes.*

The condition on the value of H at the ending point is necessary by Proposition 5.

Proof. We shall extend the function H to every interior vertex $x \in B$ by making

$$H(x) = \min_{v \in \pi}\{H(v) + d(v, x)\}. \tag{2}$$

By Proposition 2, it is sufficient to check that the first axiom of the height function definition is satisfied. This will be a straightforward consequence of the next two claims.

Claim: $H(y) - H(x) \le d(x, y)$ for all vertices $x, y \in B$. To prove this, consider a vertex $u_x \in \pi$ such that $H(x) = H(u_x) + d(u_x, x)$. Since $H(y) \le H(u_x) + d(u_x, y) \le H(u_x) + d(u_x, x) + d(x, y) = H(x) + d(x, y)$, it follows that $H(y) - H(x) \le d(x, y)$.

Claim: Let x and y be adjacent vertices of B such that $[x, y]$ is positively oriented. Then $H(y) \equiv H(x) + 1 \pmod p$. This is immediate from Proposition 4 and the way that H has been constructed. □

Since $\{p, \infty\}$-boards have no interior vertices, when $q = \infty$ it is not even necessary extending H to the interior of B in the proof of Theorem 6.

Theorem 6 has a converse (Corollary 8). To prove it, we need first a converse of Proposition 2.

Proposition 7. *Let B be a bicolored $\{p, q\}$-board. There exists a height function defined on B if B can be tiled by $\{p, q\}$-dominoes.*

Proof. It is by induction over the number of dominoes needed to tile B.

When B is a domino, the lifting of the boundary of B defines a height function over B.

Now we suppose that every board tiled by $n - 1$ dominoes can be provided with a height function. Let B be a board tiled by n dominoes. We cut B along a grid-path w, in such a way that w divides B into two regions: one domino D and a board B' tiled by $n - 1$ dominoes. Then, the height function defined over B' can be extended to B naturally. □

Corollary 8 (Thurston)**.** *Let B be a bicolored $\{p, q\}$-board. We select arbitrarily one boundary vertex to be the starting and ending point of a simple closed grid-path π which describes the boundary of B. Consider the lifting H of π, and suppose that the value of H at the ending point of π is equal to zero. If $H(v) - H(u) \le d(u, v)$ is not satisfied for any pair of vertices u, v in π, then there is no tiling of B by $\{p, q\}$-dominoes.*

Proof. If $H(v) - H(u) \le d(u, v)$ is not satisfied for any pair of vertices u, v in π, there is no height function h on B by Proposition 3. The result follows from Proposition 7. □

3 Algorithm

Identity (2) provides an algorithmic solution to the tiling-by-dominoes problem. This algorithm has the board B to be tiled for initial data. The first step to do is to lift an arbitrary simple closed grid-path π representing the boundary of B. If the final value of the lifting does not correspond to the initial value then a tiling does not exist by Proposition 5. If the lifting of π gives a function H, an extension of H over the interior vertices is defined using identity (2). This can be done recursively, beginning with all the boundary vertices whose height $h_0 \in \mathbb{Z}$ is the smallest, following the positively oriented edges from these vertices given all vertices of height $h_0 + 1$ at once. Then we obtain all the vertices of height $h_0 + 2$ from those of height $h_0 + 1$ in the same way, but it is necessary to examine the new and previous values on the boundary looking for inconsistencies: if the previous height agrees with or is less than the new one, it is left the previous and the process continues, but if the previous height is greater than the new one, a tiling is impossible by Corollary 8. The process is continued until all vertices are covered, or until we find an inconsistency in the height values, which proves impossibility.

4 Examples

4.1 Figure 4 revisited

The implementation of height functions to find the tiling of Figure 4 is illustrated in Figure 5. The bicolored board contains 5 white and 5 black pentagons. By Proposition 5 a lifting of the boundary gives a function H defined over the boundary vertices of the board. After that, a height function over the interior vertices is defined successfully by using the identity (2). Finally, a tiling by $\{5, 4\}$-dominoes is obtained by deleting the edges whose ends have difference 4.

4.2 Figure 1 revisited

Figure 6 is a hyperbolic variant of Figure 1. It is clear that a tiling by dominoes is impossible. Notice that the height function on the boundary does not satisfy identity (1).

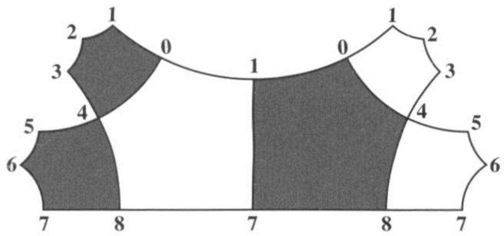

Figure 6: In this hyperbolic board, the adjacent vertices with height 1 and 7 do not satisfy identity (1).

5 About the birth of height functions: Cayley graphs

Conway and Lagarias [2] introduced Conway's tiling groups which give a necessary (but not sufficient) condition for a domain to be tileable. The technique allows us to address Euclidean tiling problems where the board is as above (i.e., a $\{p, q\}$-board whose boundary is a simple closed grid-path on a grid $\{p, q\}$) but the tiles can be formed by more than two p-gons, being its boundaries simple closed grid-paths. The case $\{6, 3\}$ also is allowed, i.e., it is not necessary to have a bicolored grid.

Each tiling problem consists of three data: a grid, the tiles, and a board. Below we summarize the use of each data in the method of Conway.

Grid It determines a group F which describes all the grid-paths starting at a given vertex. When q is even, F is the free group with $q/2$ generators. For example, the words in $F = \langle X, Y \rangle$ describe grid-paths in the grid $\{4, 4\}$: symbols X, X^{-1}, Y and Y^{-1} are associated respectively with one horizontal step to the right, one horizontal step to the left, one vertical step upward and one vertical step down. On the other hand, the words in $F = \langle a_0, a_1, a_2 : a_0^2 = a_1^2 = a_2^2 = e \rangle$ describe all the grid-paths in the grid $\{6, 3\}$ starting at a given vertex: symbol a_k is associated with one step in a direction parallel to the unit vector $(\cos(2\pi k/3), \sin(2\pi k/3))$, for $k = 0, 1, 2$.

Tiles Each tile T determines a word $W \in F$ obtained by traveling along its boundary. *Conway's tiling group* G is the quotient of F by the relations describing the tiles, that is, $G = F/\langle W_1, \dots, W_k \rangle$, where W_j is the word corresponding to the tile T_j. For example, Conway's tiling group for the dominoes problem in the grid $\{4, 4\}$ is

$$G = \langle X, Y : X^2YX^{-2}Y^{-1} = Y^2XY^{-2}X^{-1} = e \rangle. \tag{3}$$

Notice that G is well defined since a change of the starting point to travel the boundary of T gives rise to a conjugate word, and going around T in the other direction (clockwise or counterclockwise) gives rise to an inverse word.

Board The perimeter of the board B also gives a word W_0. *Conway's criterion* says: *If B can be tiled by tiles $T_1, \dots, T_k$, then $W_0 = e$ in G.* The proof of this result is quite easy; one can try by induction over the number of tiles, for instance.

The Cayley graph $\Gamma(G)$ of G is a resource commonly used to analyze whether $W_0 = e$.

For the Euclidean cases $\{4, 4\}$ and $\{3, 6\}$, Thurston [7, Section 4] had the idea of embedding $\Gamma(G)$ in $\mathbb{R}^3$, thereby obtaining an algorithm that quickly decide whether a given $\{p, q\}$-board is tileable by dominoes. For example, when G is given by (3) the vertices of $\Gamma(G)$ are the points $(x, y, z) \in \mathbb{R}^3$ for which

$$z \underset{(\text{mod } 4)}{\equiv} \begin{cases} 0 \text{ if } x \text{ and } y \text{ are both even,} \\ 1 \text{ if } x \text{ is odd and } y \text{ is even,} \\ 2 \text{ if } x \text{ and } y \text{ are both odd,} \\ 3 \text{ if } x \text{ is even and } y \text{ is odd.} \end{cases}$$

There is an edge of $\Gamma(G)$ joining the vertices $u, v \in \Gamma(G)$ just when $|u - v| = \sqrt{2}$ (see Figure 7). The orthogonal projection of $\Gamma(G)$ to the xy-plane maps edges of $\Gamma(G)$ onto edges of the square grid (see Figure 8). A 2-complex $\Gamma^2(G)$ is defined by gluing hexagons onto $\Gamma(G)$; each hexagon corresponds to a domino by the orthogonal projection (see Figure 9). A lifting of a tiled board B is a continuous inverse of the orthogonal projection, defined on B. *This is just a height function.*

Hence a fixed tiling of B lifts to a surface $S_0 \subset \Gamma^2(G)$ such that the orthogonal projection $S_0 \to B$ is a bijection. In fact, the identity (2) and the above algorithm based on it were used by Thurston to produce a surface $\Sigma \subset \Gamma^2(G)$ which is the lowest among all surfaces $S \subset \Gamma^2(G)$ satisfying two conditions:

1. the orthogonal projection $S \to B$ is a bijection,
2. $\partial S = \partial S_0$.

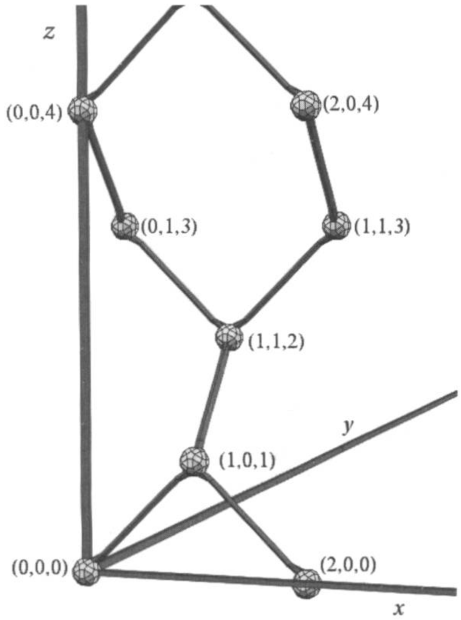

Figure 7: A small portion of the Cayley graph $\Gamma(G)$.

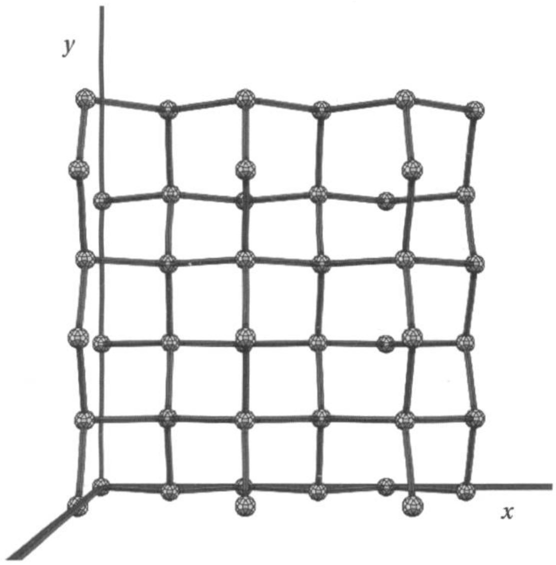

Figure 8: Perspective view of $\Gamma(G)$ from above.

References

[1] F. BONAHON: *Low-Dimensional Geometry: From Euclidean Surfaces to Hyperbolic Knots*. Student Mathematical Library, Vol. 49, AMS-IAS, 2009.

[2] J. CONWAY, J. LAGARIAS: *Tiling with polyominoes and combinatorial group theory*. J. Combin. Theory. Ser. A **52**, 183–208 (1990).

[3] T. CHABOUD: *Domino tiling in planar graphs with regular and bipartite dual*. Theoret. Comput. Sci. **159**, 137–142 (1996).

[4] H. COHN, R. KENYON, J. PROPP: *A variational principle for domino tilings*. J. Amer. Math. Soc. **14**, no. 2, 297–346 (2001).

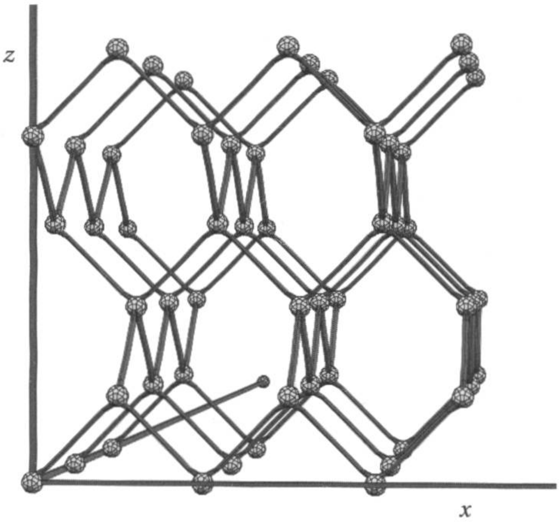

Figure 9: Some hexagons which are projected to dominoes.

[5] J.C. FOURNIER: *Pavage des figures planes sans trous par des dominos: fondement graphique de l'algorithme de Thurston, parallélisation, unicité et décomposition*. Theoret. Comput. Sci. **159**, 105–128 (1996).

[6] R. KENYON: *Lectures on dimers*. Statistical Mechanics. S. Sheffield and T. Spencer (Eds.) IAS/Park City Math. Ser. Vol. 16. Amer. Math. Soc. 191–230, 2009.

[7] W. THURSTON: *Conway's tiling groups*. Amer. Math. Monthly. **97**, no. 8, 757–773 (1990).

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