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Game show shenanigans: Monty Hall meets mathematical logic

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1 Introduction

The classical *Monty Hall problem* stipulates that a hypothetical game show contestant be presented with three doors and told that behind one door is a car and behind the other two doors are goats (the problem is loosely based on the American television game show *Let's Make a Deal* and named after the show's original host). The player is to choose a door, and then Monty (the host) opens a different door which contains a goat (such a door exists regardless of the contestant's choice, and the contestant knows in advance that Monty will do this). Then the player has the option of either keeping the door she chose or picking a different door. Subsequently, Monty opens the chosen door and the contestant wins the

Beim klassischen Monty-Hall-Problem oder Ziegenproblem sind in einer Quizshow hinter drei Türen zufällig ein Auto (als Hauptgewinn) und zwei Ziegen (als Nieten) verborgen. Der Kandidat wählt eine Tür. Der Moderator, der weiss was sich hinter jeder Tür verbirgt, öffnet daraufhin eine andere als die gewählte Tür und zwar eine mit einer Ziege dahinter. Der Kandidat darf dann entweder bei seiner Wahl bleiben oder noch wechseln. Es ist leicht zu sehen, dass der Kandidat durch Wechseln der gewählten Tür seine ursprüngliche Gewinnchance von 1/3 auf 2/3 erhöht. Die Autoren der vorliegenden Arbeit schlagen nun zwei Varianten vor, wie dieses Spiel modifiziert werden kann, damit es für beide Seiten (Kandidat und Veranstalter der Quizshow) attraktiver wird: Die Strategie des Kandidaten seine Gewinnchance zu erhöhen soll schwieriger sein, und die Gewinnchance bei optimaler Strategie soll kleiner werden.

prize which lies behind it. It is not hard to show that switching is the optimal strategy, with probability of winning jumping from 1/3 to 2/3. Indeed, the only way the contestant loses by switching is if she originally picked the car. Since she had only a 1/3 chance of picking the car, she has only a 1/3 chance of losing the game by switching. Therefore, she has a 2/3 chance of winning the car by changing her selection.

The above Monty Hall game has been utilized regularly by the game show *Let's Play a Game*, whose host (coincidentally) is also named Monty. Before Marilyn vos Savant published the ideal strategy for winning (namely, to always switch) in *Parade Magazine* (to be clear, the solution actually did appear in the magazine on September 9, 1990), the vast majority of the contestants didn't realize it was to their advantage to change their selection, and only 35% of the games ended with a contestant driving home a new car. However, after Marilyn's *Parade* column, the contestants found the car 68% of the time. Needless to say, this didn't please the executives at *Let's Play a Game*.

The show recently contracted you, a mathematician, to retool the Monty Hall game so that (a) playing with optimal strategy is more difficult, and (b) the probability of winning with optimal strategy is greater than $\frac{1}{2}$ (so as not to incur the wrath of the Game Show Contestant Union) but less than $\frac{2}{3}$. The show's budget for research and development is low, and so they want you to construct a game which, as in the original, requires a contestant to pick among 3 doors with prizes hidden behind (some subset of) them. After brainstorming for a couple of weeks, you have managed to come up with the following variants:

Game One: **Truth Triad.** A contestant is presented three doors labeled 1, 2, and 3. Further, the contestant is informed that behind one door is \$20,000, behind the other two doors is motor oil, and that all three doors are equally likely to contain the money. A door number is determined at random (but not told to the contestant) to house the cash. Monty (the host), who knows where the green is, randomly chooses a proposition φ concerning the location of the cash (this will be made precise shortly). He then presents φ to the contestant. The contestant, after a reasonable period of time to analyze φ, is presented with the truth value of φ (true or false, relative to the randomly chosen door). She then chooses a door, after which Monty opens the chosen door and the contestant wins the prize which lies behind it.

Game Two: **Full Monty.** A contestant is presented three doors labeled 1, 2, and 3 as before. But now, there are no restrictions placed on how many doors have \$20,000 behind them, and all possibilities are equally likely (so, for example, it is possible that all doors house motor oil, and at the other extreme, that \$20,000 is behind every door). This gives 8 equally likely outcomes, and the contestant is made aware of this fact in advance. One of the 8 prize configurations is chosen at random, but not revealed to the player. Then Monty randomly chooses a proposition φ concerning the location of the money along with its truth value (again, relative to the prize configuration). After being given a period of time to ponder the proposition and its truth value, she chooses a door. As before, Monty opens the chosen door and the contestant wins the prize which lies behind it.

Excited about your new ideas, you get to work solving the following problems: determine the optimal strategy for winning the games (i.e., finding money) and, using said strategy, determine the probability that the contestant will win.

2 Propositional Preliminaries

Now that you have come up with the general idea for two games, it is time to make their descriptions more precise. What is meant by "a proposition concerning the location of the money?" You quickly realize that there is a setting which is perfectly suited to give a formal answer to this question, namely, propositional logic.

The particular language $\mathcal{L}$ you have in mind contains three sentence symbols $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$. The symbol $\mathbf{A}_1$ models the assertion, "Money is behind door 1." Analogous interpretations are given to $\mathbf{A}_2$ and $\mathbf{A}_3$. Further, $\mathcal{L}$ contains the usual logical connectives $\neg$ (*not*), $\vee$ (*or*), $\wedge$ (*and*), $\rightarrow$ (*implies*), and $\leftrightarrow$ (*if and only if*) as well as punctuation (,) for unique readability. The set $\mathcal{F}$ of formulas is generated from the sentence symbols using the propositional connectives in the familiar way. We pause to give a few simple examples of formulas along with their intended English language translations.

Example 1. We have the following:

1. *formula*: $\mathbf{A}_2$; *translation*: "Money is behind door 2."
2. *formula*: $\mathbf{A}_2 \vee \mathbf{A}_3$; *translation*: "Money is behind door 2 or door 3."
3. *formula*: $(\mathbf{A}_1 \vee \mathbf{A}_3) \wedge \neg\mathbf{A}_2$; *translation*: "Money is behind door 1 or door 3 but not behind door 2."
4. *formula*: $\mathbf{A}_2 \rightarrow \neg(\mathbf{A}_1 \vee \mathbf{A}_3)$; *translation*: "If money is behind door 2, then money is not behind door 1 nor behind door 3."
5. *formula*: $\mathbf{A}_1 \leftrightarrow \neg(\mathbf{A}_2 \vee \mathbf{A}_3)$; *translation*: "Money is behind door 1 if and only if money is not behind door 2 nor behind door 3."

Next, fix a set $\{T, F\}$ consisting of two distinct elements T, called *truth*, and F, called *falsity*. Any function $\upsilon\colon \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\} \rightarrow \{T, F\}$ (henceforth called a *truth assignment*) can be uniquely extended to a function $\overline{\upsilon}\colon \mathcal{F} \rightarrow \{T, F\}$ (cf. [5], Theorem 13A). Again, we pause to present an example.

Example 2. *Suppose that* $\upsilon(\mathbf{A}_1) = T$ *and* $\upsilon(\mathbf{A}_2) = \upsilon(\mathbf{A}_3) = F$. *Then* $\overline{\upsilon}(\mathbf{A}_1) = \overline{\upsilon}(\neg\mathbf{A}_2) = \overline{\upsilon}(\mathbf{A}_1 \vee \mathbf{A}_2) = T$, *whereas* $\overline{\upsilon}(\mathbf{A}_2 \wedge \mathbf{A}_3) = \overline{\upsilon}(\mathbf{A}_1 \rightarrow \mathbf{A}_2) = \overline{\upsilon}(\mathbf{A}_1 \leftrightarrow \mathbf{A}_2) = F$.

A simple yet important fact is that given any truth assignment υ and any formula φ, a truth table enables one to effectively determine whether $\overline{\upsilon}(\varphi) = T$ or $\overline{\upsilon}(\varphi) = F$.

Finally, you have the tools needed to formalize "a proposition φ concerning the location of the money": φ is simply a formula in the language $\mathcal{L}$ defined above. You want Monty to choose φ at random (i.e., you want all propositions to be equally likely to be picked by Monty). This poses a problem for multiple reasons, but the one with which you are most concerned is the practical one:

Problem 1. *There are an infinite number of formulas, so in practice, Monty simply cannot pick one at random!*

So now you're stuck... but is there a way out? You notice that although there are indeed infinitely many formulas, there are large groups of them which make the same assertions.

For example, the formulas in $S := \{\mathbf{A}_1, \mathbf{A}_1 \vee \mathbf{A}_1, (\mathbf{A}_1 \vee \mathbf{A}_1) \vee \mathbf{A}_1, \ldots\}$ are, in a certain way, all equivalent. But how does one make this precise? They are equivalent in the following sense: if υ is any truth assignment such that $\overline{\upsilon}(\alpha) = T$ for *some* $\alpha \in S$, then $\overline{\upsilon}(\beta) = T$ for *all* $\beta \in S$. From this observation, you are lead naturally to the relation $\sim$ defined on the set $\mathcal{F}$ of all formulas by $\alpha \sim \beta$ if and only if $\overline{\upsilon}(\alpha) = \overline{\upsilon}(\beta)$ for every truth assignment υ. It is a simple matter to verify that $\sim$ is an equivalence relation on $\mathcal{F}$ (the reader may notice that for any formulas α and β, $\alpha \sim \beta$ if and only if $\alpha \leftrightarrow \beta$ is a tautology; using logical nomenclature, $\alpha \sim \beta$ if and only if $\alpha \models\dashv \beta$).

Now, if the set of equivalence classes happens to be *finite*, then you can simply give a complete set of representatives of $\mathcal{F}$ (modulo $\sim$) to Monty and he can randomly choose one without any loss of expressiveness. Problem solved! It remains to determine if, in fact, there are but finitely many equivalence classes. Toward this end, you introduce the following definition.

Definition 1. *Let* $\{T, F\}^3 := \{T, F\} \times \{T, F\} \times \{T, F\}$. *A (tertiary) Boolean function is a function* $B \colon \{T, F\}^3 \to \{T, F\}$.

Let $\mathscr{B}$ denote the collection of Boolean functions. It follows from basic combinatorics that

$$|\mathscr{B}| = 2^{(2^3)} = 256. \tag{2.1}$$

Moreover, every formula φ determines a Boolean function B_φ as follows: let $(x_1, x_2, x_3) \in \{T, F\}^3$. Then $B_\varphi((x_1, x_2, x_3))$ = the truth value of φ when $\mathbf{A}_1$ is assigned x_1, $\mathbf{A}_2$ is assigned x_2, and $\mathbf{A}_3$ is assigned x_3. More formally, let $\mathscr{T}$ denote the collection of all truth assignments $\upsilon \colon \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\} \to \{T, F\}$. Then $B_\varphi((\upsilon(\mathbf{A}_1), \upsilon(\mathbf{A}_2), \upsilon(\mathbf{A}_3)) := \overline{\upsilon}(\varphi)$, where υ varies over $\mathscr{T}$. We pause to present a concrete example.

Example 3. *Let* $\varphi := (\mathbf{A}_1 \vee \mathbf{A}_2) \wedge \mathbf{A}_3$. *Then*

1. $B_\varphi((T, T, T)) = B_\varphi((F, T, T)) = B_\varphi((T, F, T)) = T$, *and*
2. $B_\varphi((T, T, F)) = B_\varphi((F, F, F)) = B_\varphi((F, F, T)) = B_\varphi((T, F, F))$ $= B_\varphi((F, T, F)) = F$.

Now, it follows immediately from the definition of $\sim$ and B_φ that for any formulas α, $\beta \in \mathcal{F}$,

$$\text{([5], Theorem 15A, part (b))}\ \alpha \sim \beta \text{ if and only if } B_\alpha = B_\beta. \tag{2.2}$$

For $\varphi \in \mathcal{F}$, let $[\varphi] := \{\beta \in \mathcal{F} : \varphi \sim \beta\}$ be the equivalence class of φ modulo $\sim$. Then (2.2) implies that the function $f : \mathcal{F}/{\sim} \to \mathscr{B}$ defined by

$$f([\varphi]) := B_\varphi \tag{2.3}$$

is well defined and one-to-one. Our first proposition is perhaps less obvious:

Proposition 1 ([5], Theorem 15B). *Every Boolean function is of the form* B_γ *for some formula* γ.

Sketch of Proof. Let B be an arbitrary Boolean function, and let $k := |B^{-1}(T)|$. We consider several cases.

Case 1: $k = 8$. Then $B(\vec{x}) = T$ for all $\vec{x} \in \{T, F\}^3$. In this case, take $\gamma := \mathbf{A}_1 \vee \neg\mathbf{A}_1$.

Case 2: $k = 0$. Then $B(\vec{x}) = F$ for all $\vec{x} \in \{T, F\}^3$. Now choose $\gamma := \mathbf{A}_1 \wedge \neg\mathbf{A}_1$.

Case 3: $1 \leq k \leq 7$. List all vectors $\vec{x}$ for which $B(\vec{x}) = T$ as follows:

$$\begin{aligned}\vec{x}_1 &:= (x_{11}, x_{12}, x_{13})\\ \vec{x}_2 &:= (x_{21}, x_{22}, x_{23})\\ &\vdots\\ \vec{x}_k &:= (x_{k1}, x_{k2}, x_{k3}).\end{aligned}$$

Now, for $1 \leq i \leq k$ and $1 \leq j \leq 3$, define the formula α_{ij} by

$$\alpha_{ij} = \begin{cases} \mathbf{A}_j & \text{if } x_{ij} = T, \\ \neg\mathbf{A}_j & \text{if } x_{ij} = F. \end{cases}$$

Finally, set $\beta_i := \alpha_{i1} \wedge \alpha_{i2} \wedge \alpha_{i3}$, and let $\gamma := \beta_1 \vee \cdots \vee \beta_k$. It is not hard to check that $B = B_\gamma$, completing the proof. □

Proposition 1 allows us to conclude that the function f defined in (2.3) is onto. Therefore, there are exactly 256 equivalence classes of formulas modulo $\sim$. Not only this, but the previous proof yields a nice algorithm for choosing a representative from each class. As an added bonus, the proof shows how to do this in such a way that each chosen formula has a relatively simple form. To be fair to the contestant and since the show has a limited amount of time to allow the contestant to deliberate, you are quite happy with your findings. In particular, you have managed to resolve Problem 1. Having laid the ground work, you are excited to move on and calculate the probability of winning the new games you created (and since it would be nice to get paid, you cross your fingers and hope that the probability of winning is between $\frac{1}{2}$ and $\frac{2}{3}$).

3 Truth Triad

We begin by reminding the reader of the description of Truth Triad: Mary Contestant is presented three doors labeled 1, 2, and 3. Further, Mary is informed that behind one door is \$20,000 and behind the other two doors is motor oil, and that the money is equally likely to be behind each of the three doors. A door number is determined at random (but not told to Mary) and the \$20,000 is placed behind the chosen door. Monty, who knows the location of the money, randomly chooses a proposition φ (in the language $\mathcal{L}$ introduced in the previous section). He then presents φ to Mary. Mary, after a reasonable period of time to analyze φ, is given the truth value of φ (relative to the door chosen to house the money). She then must choose a door, after which Monty opens the chosen door and then Mary wins the prize which lies behind it.

After setting up the logical machinery in Section 2, you are now able to devise an optimal strategy for Mary and determine the probability that she wins the money using this strategy. Note first that the rules of the game allow exactly 3 possible truth assignments (hence the game's moniker) v_1, v_2, and v_3 defined as follows:

$$v_1(\mathbf{A}_1) = T, \quad v_1(\mathbf{A}_2) = v_1(\mathbf{A}_3) = F, \tag{3.1}$$

$$v_2(\mathbf{A}_2) = T, \quad v_2(\mathbf{A}_1) = v_2(\mathbf{A}_3) = F, \tag{3.2}$$

$$v_3(\mathbf{A}_3) = T, \quad v_3(\mathbf{A}_1) = v_3(\mathbf{A}_2) = F. \tag{3.3}$$

Recall, moreover, that Mary is told in advance that exactly one door is to contain the money, so she knows that the above truth assignments are the only ones possible. Further, if Mary is given any proposition φ (from the set of 256 propositions you've given to Monty), she can effectively determine via a truth table which of the above three truth assignments make φ true, if any.

Finally, you are equipped to analyze the game. First, since truth assignments (3.1)–(3.3) are equally likely, it is easy to see that for any proposition φ, the probability $\mathbb{P}_{\mathcal{T}}(\varphi)$ that φ is true (relative to a randomly chosen assignment from (3.1)–(3.3) above) is

$$\begin{aligned} \mathbb{P}_{\mathcal{T}}(\varphi) &= \frac{\text{the number of truth assignments in (3.1)–(3.3) making } \varphi \text{ true}}{\text{the number of truth assignments in (3.1)–(3.3)}} \\ &= \frac{|B_{\varphi}^{-1}(T) \cap \{(T, F, F), (F, T, F), (F, F, T)\}|}{3}. \end{aligned} \tag{3.4}$$

It is clear from (3.4) that

$$\text{for any formula } \varphi, \mathbb{P}_{\mathcal{T}}(\varphi) \in \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}. \tag{3.5}$$

For a randomly selected formula φ and for $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, let $\Pr(\mathbb{P}_{\mathcal{T}}(\varphi) = \alpha)$ denote the probability that $\mathbb{P}_{\mathcal{T}}(\varphi) = \alpha$. Then

$$\Pr(\mathbb{P}_{\mathcal{T}}(\varphi) = 0) = \Pr(\mathbb{P}_{\mathcal{T}}(\varphi) = 1) = 1/8, \tag{3.6}$$

and

$$\Pr(\mathbb{P}_{\mathcal{T}}(\varphi) = 1/3) = \Pr(\mathbb{P}_{\mathcal{T}}(\varphi) = 2/3) = 3/8. \tag{3.7}$$

To verify (3.6), you begin by simply counting the number of formulas φ (again, from the 256 that you gave to Monty earlier) for which $\mathbb{P}_{\mathcal{T}}(\varphi) = 0$. But this amounts to counting the number of Boolean functions B for which $B(T, F, F) = B(F, T, F) = B(F, F, T) = F$. Elementary combinatorics yields $2^5 = 32$ such functions. Thus $\Pr(\mathbb{P}_{\mathcal{T}}(\varphi) = 0) = \frac{32}{256} = \frac{1}{8}$. One shows analogously that $\Pr(\mathbb{P}_{\mathcal{T}}(\varphi) = 1) = \frac{1}{8}$. A similar argument establishes (3.7). Alternatively, note that if one restricts the domain of the Boolean functions to $\{(T, F, F), (F, T, F), (F, F, T)\}$, then one obtains but 8 different functions, and (3.6), (3.7) follow.

It is now an hour before Mary is to appear on *Let's Play a Game*. One of the doors is randomly picked, and the show's staff puts \$20,000 behind the chosen door and motor oil behind the other two. Monty randomly chooses a proposition φ to present to Mary, and then Mary is brought on stage and given φ. She is allotted several minutes to analyze φ, after which time Monty reveals φ's truth value (relative to the randomly selected prize configuration) and then asks Mary to choose a door. Again, we remind the reader that Mary knows that the money is behind exactly one of the doors. You now consider the possible cases systematically.

Case 1: $\mathbb{P}_{\mathcal{T}}(\varphi) = 0$. This is somewhat unfortunate for Mary in that even after Monty reveals the truth value of φ, Mary gains nothing. She can compute (via a truth table, if she is fast enough) that $\overline{v}_i(\varphi) = F$ for all $i \in \{1, 2, 3\}$ (recall that v_i is defined in (3.1)–(3.3)). Since she knows that exactly one of the three truth assignments (1)–(3) is to be realized, she doesn't *need* Monty to tell her that φ is false (even though he will); she already figured this out from the truth table. Said another way, the assumption that the money is behind exactly one of the three doors implies $\neg\varphi$, so she has gained no new information. Thus all Mary can do is pick her favorite integer between 1 and 3, and the probability of winning is $1/3$.

Case 2: $\mathbb{P}_{\mathcal{T}}(\varphi) = 1$. The analysis proceeds exactly as in Case 1, and again, the probability of winning is $\frac{1}{3}$.

Case 3: $\mathbb{P}_{\mathcal{T}}(\varphi) = \frac{1}{3}$. Then there is a unique $i \in \{1, 2, 3\}$ such that $\overline{v}_i(\varphi) = T$. Moreover, by a truth table, Mary can find i. Now, if Monty tells her that φ is true, then Mary wins: the money is behind door i. If Monty tells her that φ is false, then Mary knows the money is not behind door i, and thus the probability Mary will find the cash is $\frac{1}{2}$. Therefore, the probability that Mary will find the \$20,000 in this case is $\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{3}$.

Case 4: $\mathbb{P}_{\mathcal{T}}(\varphi) = \frac{2}{3}$. The analysis proceeds as in Case 3, and the probability of Mary of winning is $\frac{2}{3}$ in this case as well.

At long last, you have enough data to compute the probability of Mary finding the money.

Theorem 1. *The probability $\mathbb{P}_1$ that a contestant playing Truth Triad with optimal strategy will win is $\frac{7}{12}$ (or $58.\overline{3}\%$).*

Proof. By (3.6), (3.7), and Cases 1–4 above, the probability that a contestant playing with optimal strategy will win is $\mathbb{P}_1 = \frac{1}{8} \cdot \frac{1}{3} + \frac{1}{8} \cdot \frac{1}{3} + \frac{3}{8} \cdot \frac{2}{3} + \frac{3}{8} \cdot \frac{2}{3} = \frac{1}{12} + \frac{1}{2} = \frac{7}{12}$. □

4 Full Monty

As we did with Truth Triad in the previous section, we quickly review Full Monty for the reader. Mary is presented three doors labeled 1, 2, and 3. But now there are no restrictions placed on how many doors house the money, and all possibilities are equally likely. This gives 8 equally likely outcomes, and Mary is made aware of this in advance. One of the 8 configurations is chosen at random, but not revealed to her. Then Monty randomly chooses a proposition φ and presents it to Mary along with its truth value (relative to the randomly chosen prize configuration). Again, she is informed in advance that this will be done. After being given a period of time to ponder the proposition and its truth value, she chooses a door. Then Monty opens the chosen door and she wins the prize which lies behind it. Note that there is no loss of generality in assuming that Monty gives Mary a *true* proposition relative to the randomly chosen prize configuration, since a false proposition can be negated (by us and by Mary). *Henceforth, we shall assume that Mary is always given a true proposition by Monty and, moreover, that Mary knows in advance that she will be given a true proposition.*

Before beginning our analysis, a few remarks are in order. Recall that in Truth Triad, Mary starts off (before Monty's help, that is) with a $33.\overline{3}\%$ probability of finding the money. With Monty's assistance, the probability increases by 25%. In Full Monty, it is easy to see that Mary begins with a 50% chance of finding money. However, Truth Triad endows Mary with more knowledge than does Full Monty. Specifically, she knows that money will be behind exactly one door. Playing Full Monty, Mary goes in blind. Therefore, you don't expect Monty to be as helpful to Mary as he was in the first game. After the smoke clears, will Mary be better off playing Truth Triad or Full Monty? Let's find out.

Your first goal is to determine Mary's optimal strategy for finding money (note that we omit the article "the" preceding "money" because money may lie behind multiple doors). Toward this end, you set up some notation. As before, for a proposition φ, let B_φ denote the corresponding Boolean function. For $1 \leq i \leq 3$, let $\pi_i : \{T, F\}^3 \to \{T, F\}$ be projection onto the ith coordinate.

Describing the optimal strategy is fairly straightforward. Suppose that it has been randomly determined which doors (if any) house money. The formal model is simply that a member of $\{T, F\}^3$ is chosen at random, and for $1 \leq i \leq 3$, there is a T in the ith coordinate if and only if there is money behind the ith door. Now, Monty picks a formula φ that is true relative to the chosen configuration (formally, if $\vec{x} \in \{T, F\}^3$ is picked at random, then φ is also chosen randomly subject only to $B_\varphi(\vec{x}) = T$). Mary may construct a truth table to find the function B_φ. For a fixed formula φ and $1 \leq i \leq 3$, let $\Pr_\varphi(i)$ denote the probability that money lies behind door i (given φ). Then it is easy to see that

$$\begin{aligned}\Pr_\varphi(i) &= \frac{\text{the number of members of } B_\varphi^{-1}(T) \text{ with a } T \text{ in the } i\text{th coordinate}}{\text{the cardinality of } B_\varphi^{-1}(T)} \\ &= \frac{|B_\varphi^{-1}(T) \cap \pi_i^{-1}(T)|}{|B_\varphi^{-1}(T)|}.\end{aligned} \tag{4.1}$$

Thus Mary's best strategy is to choose $i \in \{1, 2, 3\}$ such that $|B_\varphi^{-1}(T) \cap \pi_i^{-1}(T)|$ is a maximum (note that there may be multiple such i).

Now that you have determined the optimal strategy for winning the game, you turn your attention to the global problem of finding the probability that Mary Contestant will win \$20,000 using the strategy just described. You have a nice insight into how to transform this probability problem into an elementary linear algebra problem. *Toward this end, in what follows, we denote F by* 0 *and T by* 1. This conversion allows you to represent Boolean functions as certain 8×4 binary matrices (that is, as matrices all of whose entries are either 0 or 1). The details follow.

Let

$$\mathbf{x} := \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y} := \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{z} := \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then notice that for any formula φ, the Boolean function B_φ can be represented by the 8×4 matrix $[\mathbf{x}\,\mathbf{y}\,\mathbf{z}\,\mathbf{b}^\varphi]$, where for each i, $1 \le i \le 8$, $b_i^\varphi := B_\varphi(x_i, y_i, z_i)$. It should now be clear to the reader how the definitions of **x**, **y**, and **z** came into being: the set of rows of the matrix $[\mathbf{x}\,\mathbf{y}\,\mathbf{z}]$ is precisely the domain of the Boolean function B_φ for any formula φ. Let us agree to call the matrix $M_\varphi := [\mathbf{x}\,\mathbf{y}\,\mathbf{z}\,\mathbf{b}^\varphi]$ the *Boolean matrix of* φ (matrices whose entries are all either 0 or 1 are often called Boolean matrices, (0, 1) matrices, or logic matrices). Let $\mathcal{F}_{MH}$ denote the collection of the 256 formulas given to Monty earlier. Then

$$\{M_\varphi : \varphi \in \mathcal{F}_{MH}\} = \{[\mathbf{x}\,\mathbf{y}\,\mathbf{z}\,\mathbf{v}] : \mathbf{v} \in \{0, 1\}^8\}. \tag{4.2}$$

Moreover, if $\alpha, \beta \in \mathcal{F}_{MH}$ are distinct, then $B_\alpha \neq B_\beta$. Thus

$$\text{if } \alpha, \beta \in \mathcal{F}_{MH} \text{ are distinct, then } M_\alpha \neq M_\beta. \tag{4.3}$$

You are almost ready to determine the probability of Mary winning Full Monty. The following lemmas will be required.

Lemma 1. *Let $\varphi \in \mathcal{F}_{MH}$. The probability that Monty chooses φ and presents it to Mary (recall that we reduced to the case where Monty only presents true formulas to Mary relative to the randomly chosen prize configuration) is* $\frac{\mathbf{b}^\varphi \cdot \mathbf{b}^\varphi}{1024}$.

Proof. The original probability space for Full Monty consists of all 2048 equally likely points (x, y, z, φ) such that $(x, y, z) \in \{0, 1\}^3$ and $\varphi \in \mathcal{F}_{MH}$. Since we reduced to the case where Monty only presents true formulas to Mary relative to the randomly chosen prize configuration, we fold the probability space to consist of 1024 equally likely points (x, y, z, φ) that satisfy the condition $B_\varphi(x, y, z) = 1$, with the understanding that (x, y, z) is identified with the Boolean triple determined by the dictionary $F = 0$ and $T = 1$. Indeed, given any fixed prize configuration $(x, y, z) \in \{0, 1\}^3$, it is easy to see that there are precisely $2^7 = 128$ many $\varphi \in \mathcal{F}_{MH}$ such that $B_\varphi(x, y, z) = 1$. Now, for any $\varphi \in \mathcal{F}_{MH}$, we have that φ is a true proposition for anywhere from 0 to 8 different prize configurations (x, y, z). The actual number is given as the number of 1s in $\mathbf{b}^\varphi$, or $\mathbf{b}^\varphi \cdot \mathbf{b}^\varphi$, since for precisely $\mathbf{b}^\varphi \cdot \mathbf{b}^\varphi$ many rows of M_φ we have a prize configuration (x, y, z) such that $B_\varphi(x, y, z) = 1$. We have established that the folded probability space has 1024 equally likely points and for $\varphi \in \mathcal{F}_{MH}$, there are $\mathbf{b}^\varphi \cdot \mathbf{b}^\varphi$ prize configurations such that $B_\varphi(x, y, z) = 1$. The result now follows. □

Lemma 2. *Suppose that Monty has given the formula φ to Mary. Using optimal strategy, the probability of Mary choosing a door which houses money is equal to*

$$\frac{\max(\mathbf{x}\cdot\mathbf{b}^{\varphi},\ \mathbf{y}\cdot\mathbf{b}^{\varphi},\ \mathbf{z}\cdot\mathbf{b}^{\varphi})}{\mathbf{b}^{\varphi}\cdot\mathbf{b}^{\varphi}}.$$

Proof. The proof consists, essentially, of "translating" (4.1) to the realm of Boolean matrices. So assume that Monty has given φ to Mary. Recall from (4.1) that the probability that money lies behind door 1, given φ, is given by

$$\begin{aligned}
\mathrm{Pr}_{\varphi}(1) &= \frac{\text{the number of members of } B_{\varphi}^{-1}(T) \text{ with a } T \text{ in the 1st coordinate}}{\text{the cardinality of } B_{\varphi}^{-1}(T)} \\
&= \frac{\text{the number of rows of } M_{\varphi} \text{ whose first and last entries are 1}}{\text{the number of 1s in } \mathbf{b}^{\varphi}} \qquad (4.4)\\
&= \frac{\mathbf{x}\cdot\mathbf{b}^{\varphi}}{\mathbf{b}^{\varphi}\cdot\mathbf{b}^{\varphi}}.
\end{aligned}$$

Analogous arguments apply for doors 2 and 3, and the proof is complete. □

You are now equipped to prove your second theorem:

Theorem 2. *The probability $\mathbb{P}_2$ that a contestant playing with optimal strategy will win Full Monty is $\frac{41}{64}$ (or 64.0625%).*

Proof. The probability of winning Full Monty with optimal strategy is given by

$$\begin{aligned}
\mathbb{P}_2 &= \sum_{\varphi\in\mathcal{F}_{MH}} (\text{probability } \varphi \text{ will be chosen})(\text{probability of finding money, given } \varphi) \\
&= \sum_{\varphi\in\mathcal{F}_{MH}} \left(\frac{\mathbf{b}^{\varphi}\cdot\mathbf{b}^{\varphi}}{1024} \times \frac{\max(\mathbf{x}\cdot\mathbf{b}^{\varphi},\ \mathbf{y}\cdot\mathbf{b}^{\varphi},\ \mathbf{z}\cdot\mathbf{b}^{\varphi})}{\mathbf{b}^{\varphi}\cdot\mathbf{b}^{\varphi}}\right) \qquad \text{(by Lemmas 1 and 2)} \\
&= \frac{1}{1024} \sum_{\varphi\in\mathcal{F}_{MH}} \max(\mathbf{x}\cdot\mathbf{b}^{\varphi},\ \mathbf{y}\cdot\mathbf{b}^{\varphi},\ \mathbf{z}\cdot\mathbf{b}^{\varphi}) \\
&= \frac{1}{1024} \sum_{\mathbf{v}\in\{0,1\}^8} \max(\mathbf{x}\cdot\mathbf{v}, \mathbf{y}\cdot\mathbf{v}, \mathbf{z}\cdot\mathbf{v}) \qquad \text{(by 4.2)} \\
&= \frac{1}{1024} \times 656 \qquad \text{(this follows from a direct computation)} \\
&= \frac{41}{64},
\end{aligned}$$

concluding the argument. □

Success! You are quite happy with yourself for a job well done: both games you invented are less likely to pay dividends to the contestant than the original game, and the optimal strategies require much more of the player than simply memorizing "I should always switch." Your next task: convince the execs to give you a shot at playing for a car before they usher in your new games (and your 1995 Ford station wagon gives up the ghost).

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