

Zeitschrift: Elemente der Mathematik

Band: 72 (2017)

Heft: 2

Artikel: An effective approach for integer partitions using exactly two distinct sizes of parts

Autor: Benyahia-Tani, Nesrine / Bouroubi, Sadek / Kihel, Omar

DOI: <https://doi.org/10.5169/seals-730833>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 19.11.2024

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

An effective approach for integer partitions using exactly two distinct sizes of parts

Nesrine Benyahia-Tani, Sadek Bouroubi and Omar Kihel

Nesrine Benyahia-Tani is a Lecturer of Mathematics since 2011 at the USTHB University, Algiers, Algeria. Her interests are operations research, combinatorics, number theory and cryptography.

Sadek Bouroubi is a full Professor at USTHB University, Algiers, Algeria. He received his habilitation (Doctorat d'Etat) on operational research in 2004. His current interests include combinatorial optimization, combinatorial problems and cryptography.

Omar Kihel did his Ph.D. at Université Laval in 1996. He had postdoc positions at McGill and Concordia University and at Université Laval. He joined Brock University in 2002 as full Professor. His area of research is Number Theory.

1 Introduction

A partition of a positive integer n is a sequence of non-increasing positive integers n_1 (a_1 times), n_2 (a_2 times), \dots , n_s (a_s times), with $n_i > n_{i+1}$, that sum to n . We sometimes write such partition $\pi = (n_1^{a_1} n_2^{a_2} \cdots n_s^{a_s})$, each n_i is called part of the partition π and a_i its frequency. The partition function $p(n)$ counts the partitions of n . If we ignore some

¹This research was partly supported by L'IFORCE Laboratory

Partitionen von natürlichen Zahlen sind ein gut untersuchtes Gebiet in der kombinatorischen Zahlentheorie. Schon Euler gab die erzeugende Funktion für die Anzahl solcher Partitionen an. Von grossem Interesse sind auch Partitionen mit bestimmten Eigenschaften, etwa einer gegebenen Anzahl Summanden, mit vorgegebenem kleinsten Summanden usw. In der vorliegenden Arbeit werden Partitionen betrachtet, bei denen nur zwei verschiedene Summanden auftreten, also zum Beispiel $11 = 4 + 4 + 1 + 1 + 1$. Insbesondere finden die Autoren eine explizite Formel für die Anzahl solcher Partitionen. Diese Formel wird in einen schlanken Algorithmus gegossen, der diese Partitionen liefert. Als Nebenprodukt resultiert eine Formel für die Partitionen einer Zahl n , bei denen nur s teilerfremde Summanden auftreten.

unpublished work of G.W. Leibniz, the theory of integer partitions can find its origin in the work of L. Euler [6]. In fact, he made a sustained study of partitions and partition identities, and exploited them to establish a huge number of results in Analysis in 1748. An excellent introduction to this subject can be found in the book of G.E. Andrews [2].

Definition 1.1. Let $\pi = (n_1^{a_1} n_2^{a_2} \cdots n_s^{a_s})$ be a partition of n . We say that π is a partition into k parts with s distinct sizes if

$$\begin{cases} n = a_1 n_1 + \cdots + a_s n_s; \\ n_1 > n_2 > \cdots > n_s \geq 1; \\ a_1 + \cdots + a_s = k; \\ a_1, \dots, a_s \geq 1. \end{cases} \quad (1.1)$$

Let $t(n, k, s)$ be the number of solutions of system (1.1) and $t(n, s)$ the total number of partitions of n into s distinct sizes. Then we have

$$t(n, s) = \sum_{k=s}^{\frac{2n-s(s-1)}{2}} t(n, k, s). \quad (1.2)$$

Example 1.2. Among 27 partitions of $n = 11$ into two distinct sizes, the partitions $(7^1 1^4)$, $(4^2 1^3)$, $(3^1 2^4)$ and $(3^3 1^2)$ are the only ones which are into 5 parts.

This kind of partitions appeared for the first time in the work of P.A. MacMahon [7]. Next, E. Deutsch presented the number of partitions of n into exactly two odd sizes of parts and the number of partitions of n into exactly two sizes of parts, one odd and one even. One can find these values in the Online Encyclopedia of Integer Sequences (OEIS) [8] as A117955 for the first number, A117956 for the second one and A002133 for the number of partitions of n using only 2 types of parts.

In [3] we can find a proof of effective and non-effective finiteness theorems on $t(n, k, s)$, we can cite for example the following results:

Theorem 1.3. For $k \geq s \geq 2$ and $n \geq k + \frac{s(s-1)}{2}$, we have

$$t(n, k, s) = \sum_{i=1}^{\lfloor \frac{2n-s(s-1)}{2k} \rfloor_{k-s+1}} \sum_{j=1}^{k-s+1} t(n - ki, k - j, s - 1), \quad (1.3)$$

$$t(n, k, 2) = \sum_{i=1}^{\lfloor \frac{n-1}{k} \rfloor} \tau_{k-1 \downarrow}(n - ki), \quad (1.4)$$

where $\tau_{d \downarrow}(k)$ denotes the number of positive divisors of k less than or equal to d .

2 Main results

One of the aims of this paper is to give an explicit formula for $t(n, k, 2)$ using an effective new approach. Thus, let us consider the system:

$$\begin{cases} n = a_1 n_1 + a_2 n_2; \\ a_1 + a_2 = k; \\ n_1 > n_2 \geq 1; \\ a_1, a_2 \geq 1. \end{cases} \quad (2.1)$$

Throughout the remainder of the paper let $m = n_1 - n_2$ for any solution of system (2.1).

First of all, we introduce the following lemma to prepare the main theorem.

Lemma 2.1. *System (2.1) has integral solutions if and only if the following conditions are satisfied:*

(i) $n \equiv n_2 k \pmod{m}$,

(ii) $\max(1, \lceil \frac{n}{k} \rceil - m + \chi(k|n)) \leq n_2 \leq \lfloor \frac{n}{k} \rfloor - \chi(k|n)$,

where $\chi(k|n) = 1$ if k divides n , and 0 otherwise.

Proof. By system (2.1) we have

$$\begin{pmatrix} n_1 & n_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} n \\ k \end{pmatrix}.$$

Since $m > 0$, we get

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{m} \begin{pmatrix} n - n_2 k \\ -n + n_1 k \end{pmatrix}.$$

Then, system (2.1) has integral solutions if and only if m divides $n - n_2 k$, $n - n_2 k > 0$ and $-n + n_1 k > 0$. That is,

$$n \equiv n_2 k \pmod{m} \quad \text{and} \quad \frac{n}{k} - m < n_2 < \frac{n}{k}.$$

Dependent on whether k divides n , plus or minus 1 needs to be added, which is done by $\chi(k|n)$ and the required result holds. \square

From this lemma, we can now derive the following theorem.

Theorem 2.2. *For $k \geq 2$ and $n \geq k + 1$ let $d = \gcd(n, k)$ and for any divisor e of d let \mathfrak{J}_e be the set of pairs $(\alpha, \beta) \in \mathbb{N}^2$, such that:*

- $1 \leq \alpha \leq \lfloor \frac{n-k}{e} \rfloor$ and $\gcd(\alpha, \frac{k}{e}) = 1$,
- $\beta \equiv (\frac{n}{e}) (\frac{k}{e})^{-1} \pmod{\alpha}$ and $0 \leq \beta \leq \min(\alpha - 1, \lfloor \frac{n}{k} \rfloor - \chi(k|n))$.

Then

$$t(n, k, 2) = \sum_{e|d} \sum_{(\alpha, \beta) \in \mathcal{I}_e} \left(\left\lfloor \frac{\lfloor \frac{n}{k} \rfloor - \chi(k|n) - \beta}{\alpha} \right\rfloor - \left\lceil \frac{\max(1, \lceil \frac{n}{k} \rceil + \chi(k|n) - \alpha e) - \beta}{\alpha} \right\rceil + 1 \right).$$

Proof. Put $e = \gcd(m, k)$ and let $\alpha = \frac{m}{e}$, that is $1 \leq \alpha \leq \lfloor \frac{n-k}{e} \rfloor$ and $\gcd(\alpha, \frac{k}{e}) = 1$. By Lemma 2.1, case (i), we can see that e divides d and $n_2 \equiv \binom{n}{e} \binom{k}{e}^{-1} \pmod{\alpha}$.

Let $0 \leq \beta < \alpha$, such that $\beta \equiv \binom{n}{e} \binom{k}{e}^{-1} \pmod{\alpha}$. Then

$$n_2 = \beta + t\alpha, \quad t \in \mathbb{Z}.$$

Since $\beta \leq n_2$, then $t \in \mathbb{N}$. It follows from Lemma 2.1, case (ii), that

$$\max\left(1, \left\lceil \frac{n}{k} \right\rceil - m + \chi(k|n)\right) \leq \beta + t\alpha \leq \left\lfloor \frac{n}{k} \right\rfloor - \chi(k|n).$$

Finally, $t(n, k, 2)$ equals the number of positive integers t , such that

$$\left\lceil \frac{\max(1, \lceil \frac{n}{k} \rceil - m + \chi(k|n)) - \beta}{\alpha} \right\rceil \leq t \leq \left\lfloor \frac{\lfloor \frac{n}{k} \rfloor - \chi(k|n) - \beta}{\alpha} \right\rfloor.$$

This completes the proof. □

Remark 2.3. One nice application of Theorem 2.2 concerns Algorithm 1 which allows to generate all partitions of n using exactly two distinct sizes of parts.

We remark that Algorithm 1 runs in $O(n)$.

Example 2.4. Let $n = 22$ and $k = 8$, then $d = \gcd(22, 8) = 2$. So, we have two divisors of d , $e = 1$ and $e = 2$.

Case 1: $e = 1$.

The values of α that satisfy $1 \leq \alpha \leq 14$ and $\gcd(\alpha, 8) = 1$ are 1, 3, 5, 7, 9, 11 or 13.

1. For $\alpha = 1$, we get $\beta = 0$. The pair $(1, 0)$ is accepted and gives the values:

$$t = 1, \quad n_2 = 2, \quad n_1 = 3, \quad a_2 = 2 \text{ and } a_1 = 6,$$

and then the partition $(3^6 2^2)$.

2. For $\alpha = 3$, we get $\beta = 2$. The pair $(3, 2)$ is accepted and gives the values:

$$t = 1, \quad n_2 = 2, \quad n_1 = 5, \quad a_2 = 6 \text{ and } a_1 = 2,$$

and then the partition $(5^2 2^6)$.

Algorithm 1. Partitions into k parts with exactly two distinct sizes of parts.

Require: $k \geq 2, n \geq \max\{k, 3\}$

Ensure: Set of quadruple (n_1, a_1, n_2, a_2) ,

$d \leftarrow \gcd(n, k)$

for each divisor e of d **do**

for α **from** 1 **to** $\lfloor \frac{n-k}{e} \rfloor$ **do**

if $\gcd(\alpha, \frac{k}{e}) = 1$ **then**

$\beta \leftarrow \binom{n}{e} \binom{k}{e}^{-1} \pmod{\alpha}$

if $\beta \leq \min(\alpha - 1, \lfloor \frac{n}{k} \rfloor - \chi(k|n))$ **then**

$t_1 \leftarrow \left\lceil \frac{\max(1, \lfloor \frac{n}{k} \rfloor - \alpha e + \chi(k|n)) - \beta}{\alpha} \right\rceil$

$t_2 \leftarrow \left\lfloor \frac{\lfloor \frac{n}{k} \rfloor - \chi(k|n) - \beta}{\alpha} \right\rfloor$

for t **from** t_1 **to** t_2 **do**

$n_2 \leftarrow \beta + t\alpha$

$n_1 \leftarrow \alpha e + n_2$

$a_2 \leftarrow \left\lfloor \frac{n - n_1 k}{n_2 - n_1} \right\rfloor$

$a_1 \leftarrow k - a_2$

end for

end if

end if

end for

end for

3. For $\alpha = 5$, we get $\beta = 4 > \min(4, 2)$, then the pair $(5, 4)$ is rejected.

4. For $\alpha = 7$, we get $\beta = 1$. The pair $(7, 1)$ is accepted and gives the values:

$$t = 1, n_2 = 1, n_1 = 8, a_2 = 6 \text{ and } a_1 = 2,$$

and then the partition $(8^2 \ 1^6)$.

5. For $\alpha = 9$, we get $\beta = 5 > \min(8, 2)$, then the pair $(9, 5)$ is rejected.

6. For $\alpha = 11$, we get $\beta = 3 > \min(10, 2)$, then the pair $(11, 3)$ is rejected.

7. For $\alpha = 13$, we have $\beta = 6 > \min(13, 2)$, then the pair $(13, 6)$ is rejected.

Case 2: $e = 2$.

The values of α that satisfy $1 \leq \alpha \leq 7$ and $\gcd(\alpha, 8) = 1$ are 1, 3, 5 or 7.

1. For $\alpha = 1$, we have $\beta = 0$. The pair $(1, 0)$ is accepted and gives $1 \leq t \leq 2$. Applying Algorithm 1, we obtain two partitions corresponding to the pair $(1, 0)$; the first one is $(3^7 \ 1^1)$ for $t = 1$ and the second one is $(4^3 \ 2^5)$ for $t = 2$.

2. For $\alpha = 3$, we get $\beta = 2$. The pair $(3, 2)$ is accepted and gives the values:

$$t = 1, n_2 = 2, n_1 = 8, a_2 = 7 \text{ and } a_1 = 1,$$

and then the partition $(8^1 2^7)$.

3. For $\alpha = 5$, we get $\beta = 4 > \min(4, 2)$, the pair $(5, 4)$ is rejected.

4. For $\alpha = 7$, we get $\beta = 1$. The pair $(7, 1)$ is accepted and gives the values:

$$t = 1, n_2 = 1, n_1 = 15, a_2 = 7 \text{ and } a_1 = 1,$$

and then the partition $(15^1 1^7)$.

We get, finally,

$$t(22, 8, 2) = 7.$$

3 Partitions into distinct co-prime sizes

After having counted the number $t(n, k, s)$, it would be of considerable interest to explore the number of partitions of n into k parts with exactly s distinct co-prime sizes, which we denote by $g(n, k, s)$. Thus, let us set

$$g(n, s) = \sum_{k=s}^{\frac{2n-s(s-1)}{2}} g(n, k, s). \quad (3.1)$$

Theorem 3.1. For $k \geq s \geq 2$ and $n \geq k + \frac{s(s-1)}{2}$, we have

$$g(n, k, s) = \sum_{d|n} \mu\left(\frac{n}{d}\right) t(d, k, s), \quad (3.2)$$

where $\mu(\cdot)$ denotes the Möbius function.

Proof. Let $T(n, k, s)$ be the set of partitions of n into k parts with s distinct sizes and $G(n, k, s)$ the subset of such partitions but with s distinct co-prime sizes. We notice that, the mapping from the set $T(n, k, s)$ to $\bigcup_{d|n} G(d, k, s)$ defined by:

$$(n_1^{a_1} n_2^{a_2} \cdots n_s^{a_s}) \rightarrow \left(\left(\frac{n_1}{\delta} \right)^{a_1} \left(\frac{n_2}{\delta} \right)^{a_2} \cdots \left(\frac{n_s}{\delta} \right)^{a_s} \right),$$

is a bijection, where $\delta = \gcd(n_1, n_2, \dots, n_s)$.

Consequently, we have

$$t(n, k, s) = \sum_{d|n} g(d, k, s). \quad (3.3)$$

Hence, the result follows by using the Möbius inversion formula. \square

Remark 3.2. Since $t(d, k, s) = 0$ if $d < k + \frac{s(s-1)}{2}$, the summation in (3.2) can be extended only over all divisors d of n such that $d \geq k + \frac{s(s-1)}{2}$. For example, if we take $n = 22$ and $k = 8$, then

$$g(22, 8, 2) = \mu(2)t(11, 8, 2) + \mu(1)t(22, 8, 2).$$

It can be checked using Algorithm 1 that if $n = 11$ and $k = 8$, then $t(11, 8, 2) = 2$, such partitions are $(2^3 1^5)$ and $(4^1 1^7)$. Then, according to Example 2.4, we get $g(22, 8, 2) = 7 - 2 = 5$. The partitions in question are: $(3^7 1^1)$, $(3^6 2^2)$, $(5^2 2^6)$, $(8^2 1^6)$ and $(15^1 1^7)$.

Using Theorems 3.1 and 2.2, we can construct Table 1.

Table 1. $g(n, k, 2)$, $2 \leq k < n \leq 20$.

$n \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$g(n, 2)$
3	1																		1
4	1	1																	2
5	2	2	1																5
6	1	1	2	1															5
7	3	3	2	2	1														11
8	2	2	2	2	2	1													11
9	3	3	2	3	2	2	1												16
10	2	2	4	1	3	2	2	1											17
11	5	5	3	4	2	3	2	2	1										27
12	2	2	2	2	3	2	3	2	2	1									21
13	6	6	4	5	2	4	2	3	2	2	1								37
14	3	3	5	3	4	1	4	2	3	2	2	1							33
15	4	4	3	3	4	4	2	4	2	3	2	2	1						38
16	4	4	5	3	4	3	3	2	4	2	3	2	2	1					42
17	8	8	5	7	3	5	3	4	2	4	2	3	2	2	1				59
18	3	3	5	2	5	2	4	2	4	2	4	2	3	2	2	1			46
19	9	9	6	7	3	7	3	4	3	4	2	4	2	3	2	2	1		71
20	4	4	4	4	4	3	6	2	3	3	4	2	4	2	3	2	2	1	57

From identity (3.2) we can see that if $k \geq \lfloor \frac{n}{2} \rfloor$, then $t(n, k, 2) = g(n, k, 2)$. In the present theorem we present this observation in a more explicit form.

Theorem 3.3. For $n \geq k + 1$ and $k \geq \max\{2, \lfloor \frac{n}{2} \rfloor\}$, we have

$$t(n, k, 2) = g(n, k, 2) = \tau(n - k) - \chi(n),$$

where $\tau(n)$ denotes the number of positive divisors of n and $\chi(n) = 1$ if $n = 2k$, 0 otherwise.

Proof. Let us first notice that if $k \geq 1 + \max\{2, \lfloor \frac{n}{2} \rfloor\}$, then $k \geq \lceil \frac{n+1}{2} \rceil$, and by Identity (1.4) the result yields (see [3], Corollary 3). Let now $k = \max\{2, \lfloor \frac{n}{2} \rfloor\}$. Since the result is true for $n = 3$, we can assume $k = \lfloor \frac{n}{2} \rfloor$. Let $\pi = (n_1^{a_1} n_2^{a_2})$ be a partition of n into k parts with two distinct sizes. If n is even, then $n_2 = 1$, else $n > (a_1 + a_2) n_2 = k n_2 \geq 2 \lfloor \frac{n}{2} \rfloor = n$, a contradiction. Hence, $n - k = (n_1 - 1) a_1$, in which case $n_1 - 1$ divides $n - k$. So, for each divisor d of $n - k$, we get $n_1 = d + 1$, $a_1 = \frac{n-k}{d} > 0$ and $a_2 = k - \frac{n-k}{d} > 0$, except for $d = 1$, where $a_2 = k - \frac{n-k}{1} = 0$. Thus, the result follows.

Now, if n is odd, then $n_2 = 1$ or $(n_1, n_2) = (3, 2)$. Indeed, if $(n_2 = 2 \text{ and } n_1 \geq 4)$ or $(n_2 \geq 3)$, then $n > 3a_1 + 2a_2 = 2k + a_1 \geq 2 \lfloor \frac{n}{2} \rfloor + 1 = n$, a contradiction. In case of $n_2 = 1$, by the same argument as above, we get for each divisor d of $n - k$, $n_1 = d + 1$, $a_1 = \frac{n-k}{d} > 0$ and $a_2 = k - \frac{n-k}{d} > 0$, except for $d = 1$, where $a_2 = k - \frac{n-k}{1} < 0$, which is completed by the partition $(3^{n-2k} 2^{3k-n})$. This completes the proof. \square

Remark 3.4. As shown in the proof above, the $t(n, k, 2)$ partitions have been generated explicitly, in fact, for each divisor d of $n - k$, we have :

$$(n_1^{a_1} n_2^{a_2}) = \begin{cases} \left((d+1)^{\frac{n-k}{d}} 1^{k-\frac{n-k}{d}} \right) & \text{if } (k > \lfloor \frac{n}{2} \rfloor) \text{ or } (k = \lfloor \frac{n}{2} \rfloor \text{ and } d \neq 1); \\ (3^{n-2k} 2^{3k-n}) & \text{if } n \text{ odd, } k = \lfloor \frac{n}{2} \rfloor \text{ and } d = 1; \\ \text{Does not exist} & \text{if } n \text{ even, } k = \frac{n}{2} \text{ and } d = 1. \end{cases}$$

Acknowledgement

The authors would like to thank the referee for his constructive criticism, for pertinent comments and valuable suggestions, which significantly improve the manuscript.

References

- [1] T.M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York (1976).
- [2] G.E. Andrews and K. Eriksson, *Integer Partitions*, Cambridge University Press, Cambridge (2004).
- [3] N. Benyahia-Tani and S. Bouroubi, Enumeration of the partitions of an integer into parts of a specified number of different sizes and especially two sizes, *Journal of Integer Sequences*, **14**, Article 11.3.6 (2011).
- [4] A. Charalambos Charalambides, *Enumerative Combinatorics*, Chapman & Hall/CRC (2002).
- [5] L. Comtet, *Advanced Combinatorics*, pp. 133–175. D. Reidel Publishing Company, Dordrecht-Holland, Boston, (1974).
- [6] E. Knobloch, Leibniz on combinatorics, pp. 409–430. *Historia Math*, **1** (1974).
- [7] P.A. MacMahon, *Divisors of numbers and their continuations in the theory of partitions*, Proc. London Math. Soc., **2**, pp. 75–113 (1919).
- [8] OEIS : The Online Encyclopedia of Integer Sequences. Published electronically at <http://oeis.org>, (2008).
- [9] I. Pak, Partition bijections, a survey, *Ramanujan Journal*, **12**, 5–75 (2006).

[10] H. Rademacher, On the partition function $p(n)$, *Proc. London, Math. Soc.*, **43**, 241–254 (1937).

[11] H.S. Wilf, *Lectures on integer partitions*, unpublished, <https://www.math.upenn.edu/~wilf/>

Nesrine Benyahia-Tani
Algiers 3 University
2 Ahmed Waked Street
Dely Brahim
Algiers, Algeria
e-mail: benyahiatani@yahoo.fr

Sadek Bouroubi (*Corresponding author*)
USTHB, Faculty of Mathematics
P.B. 32 El-Alia, 16111
Bab Ezzouar
Algiers, Algeria
e-mail: sbouroubi@usthb.dz
bouroubis@yahoo.fr

Omar Kihel
Brock University
Department of Mathematics and Statistics
500 Glenridge Avenue
St. Catharines
Ontario, Canada
e-mail: okihel@brocku.ca