Short note on the proof of Ptolemy's Lemma

Autor(en): **Pauni, Dura / Wanner, Gerhard**

Objekttyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **72 (2017)**

Heft 2

PDF erstellt am: **08.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-730835

Elem. Math. 72 (2017) 80 – 82
0013-6018/17/020080-3
DOI 10.4171/EM/328

Elemente der Mathematik

Short note On the proof of Ptolemy's Lemma

Đura Paunić and Gerhard Wanner

Ptolemy's Lemma. *Let $ABCD$ be a quadrilateral inscribed in a circle with sides a, b, c, d and diagonals e, f, then (see Figure 2 (1))*

$$ac + bd = ef\,. \tag{1}$$

This lemma enabled Ptolemy to compute his famous table of chords $\left(= 2\sin\frac{\alpha}{2}\right)$ and thus initiate the developments of modern Astronomy, Geography and Trigonometry (see [1]). Its historical importance is thus beyond any doubt. When the quadrilateral is a rectangle, then (1) becomes $a^2 + b^2 = e^2$, i.e., Pythagoras' theorem.

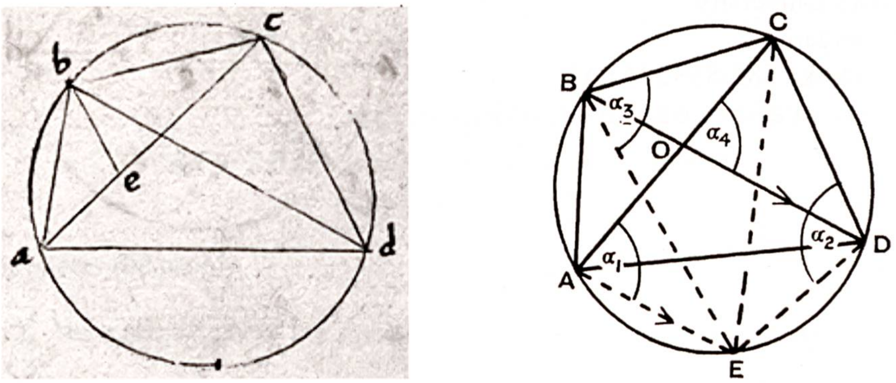

Figure 1 Ptolemy's proof in Copernicus' autograph *De revolutionibus* [4] (left); Callagy's picture from [2] (right).

Ptolemy's Proof. This proof remained the standard one for more than thousand years (see, e.g., Figure 1, left). It consisted in drawing a line *be* (notation of that picture) such that the angles *abe* and *dbc* are the same. This creates two pairs of similar triangles *abe* and *dcb* as well as *bec* and *bad*. Thales' theorem then allows the calculation of *ae* and *ec* respectively and finally to prove (1) (for more details see, e.g., [5], Section 5.1).

This proof is short and simple, but it does not reflect the geometric content of the formula, namely $ac + bd = ef$ states that the sum of the areas of two rectangles is that of another rectangle. Or, if the areas are multiplied by a constant $k = \sin\varphi$, it would be nice to *see*, without any calculation, that a parallelogram is the sum of two different parallelograms with the same angle φ. Such a proof can be made using ideas by Callagy [2] (see Figure 1, right) [1].

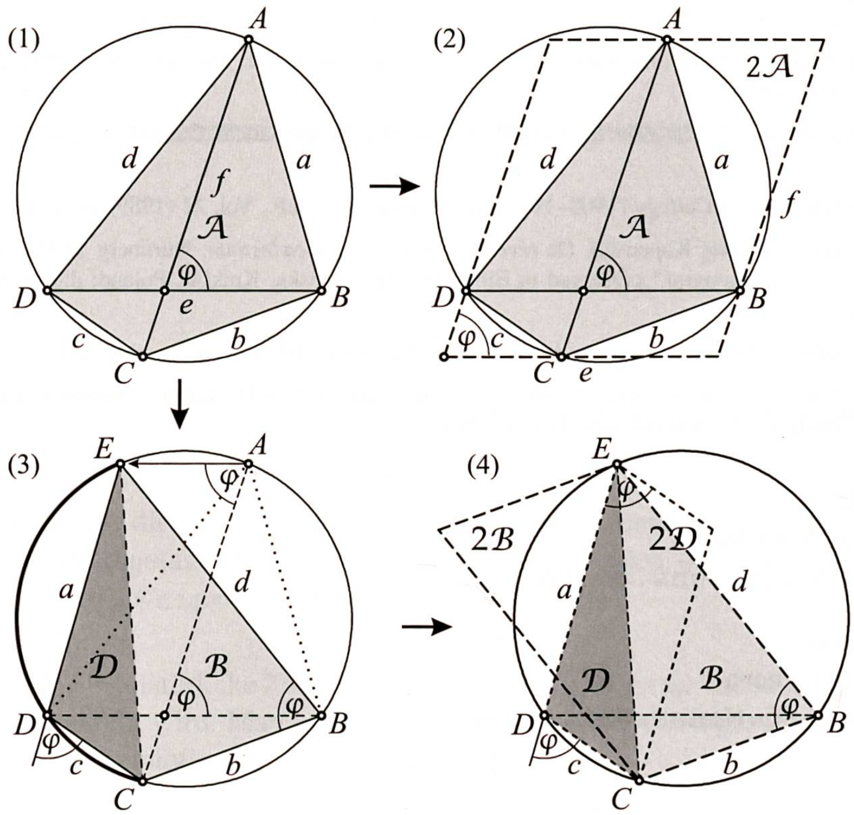

Figure 2 The parallelogram area proof in four acts.

Parallelogram area proof of Ptolemy's Lemma. First we double the area $\mathcal{A}$ of a cyclic quadrilateral $ABCD$ by circumscribing it by the parallelogram parallel to the diagonals (Callagy calls this "a well-known exercise in second year geometry", see Figure 2 (2)). The main idea is then the following: move A to E on the circle, with AE parallel to DB (Figure 2 (3)). Thus the angle φ between the diagonals is reproduced at A (Eucl. I.29), then at B (Eucl. III.21), at D (Eucl. III.22) and finally (Eucl. I.29, Figure 2 (4)) at E. The triangle EDB is mirror symmetric to ADB, with a and d exchanged. Therefore both triangles have the same area, i.e., $\mathcal{D} + \mathcal{B} = \mathcal{A}$. For the areas of the parallelograms in Figures 2 (2) and (4) we thus obtain

$$2 \cdot \mathcal{D} + 2 \cdot \mathcal{B} = 2 \cdot \mathcal{A}. \qquad (2)$$

All these parallelograms possess the same angle φ, hence (1) is just (2) divided by $k = \sin\varphi$.

Remark added in proof. Our colleague Jan Hogendijk (Univ. Utrecht) has remarked a nice connection of our proof with Ptolemy's: If in Ptolemy's picture the line *be* is extended

[1] For an obituary notice on the Irish mathematician James Callagy (1908–1988) see [3].

until the second point of intersection with the circle, one obtains precisely the line CE in our Figure 2, (3) and (4), i.e., the common diagonal of the two parallelograms. This could serve as a second motivation for the proof of Ptolemy's Lemma presented here.

References

[1] G. van Brummelen, *The Mathematics of the Heavens and the Earth, The Early History of Trigonometry*, Princeton UP (2009).

[2] J.J. Callagy, *Ptolemy's Theorem by Area Formulae*, The Mathematical Gazette, Vol. 44, No. 348 (May, 1960) p. 112.

[3] T. Christofides, *James Callagy* (1908–1988), Irish Math. Soc. Bull., Vol. 22 (1989) pp. 9–11.

[4] N. Copernicus (Mikołaj Kopernik), *De revolutionibus orbium cœlestium*, Nürnberg 1543; second impression Basel 1566; "Autograf" preserved in Biblioteka Jagiellońska, Krakòw, Poland; digitalized Neurosoft 1996.

[5] A. Ostermann, G. Wanner, *Geometry by its History*, Springer (2012).

[6] Ptolemy, *Epitoma in Almagestum Ptolemaei*, written (appr. 150 A.D., Latin (commented) translation by G. Peu[e]rbach & J. Regiomontanus, Venice 1496.

Đura Paunić
University of Novi Sad
e-mail: djura@dmi.uns.ac.rs

Gerhard Wanner
University of Geneva
e-mail: Gerhard.Wanner@unige.ch