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Short note A series representation for π

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The classical number π plays an important role not only in geometry but also in other mathematical fields, like, for example, analysis, number theory, statistics and it has interesting applications in engineering and numerous areas of physics. With regard to its relevance the properties of π have been studied thoroughly by many authors. In particular, we can find a wide variety of remarkable series, product and integral representations for π and its relatives. For detailed information on this subject we refer to the monograph Arndt and Haenel [1], the research paper Sofo [4] and the references cited therein.

It is the aim of this note to present a series representation for π which relates π to the partial sums of the Leibniz series,

$$T_k = \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \quad (k = 0, 1, 2, \dots).$$

The elegant formula

$$\frac{\pi}{4} = \lim_{k\to\infty} T_k = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots \tag{1}$$

is a well-known result. It was published for the first time in 1682 by G.W. Leibniz. Borwein et al. [2] offered an asymptotic formula fo $\pi/4 - T_k$ involving the Euler numbers and Rattaggi [3] provided various upper and lower bounds for $\pi/4 - T_k$.

Here is our representation for π.

Theorem. *We have*

$$\pi = 32 \sum_{k=0}^{\infty} (-1)^{k+1} \frac{4k^2 + 8k + 1}{(2k-1)(2k+1)(2k+3)(2k+5)} T_k^2. \tag{2}$$

Proof. First, we show that

$$a_k = b_k \quad (k = 0, 1, 2, \dots), \tag{3}$$

where

$$a_k = \frac{1}{(2k+3)^2} + \frac{1}{(2k+5)^2} - \frac{1}{2k+3} + \frac{1}{2k+5}$$

and

$$b_k = \frac{2}{2k+5}T_k^2 + \frac{4}{(2k+3)(2k+5)}T_{k+1}^2 - \frac{2}{2k+3}T_{k+2}^2.$$

Let $k \geq 0$. We have

$$\begin{aligned}
a_k &= \frac{2}{(2k+3)(2k+5)}\Big(\frac{1}{2k+3} - \frac{1}{2k+5}\Big)\\
&= \frac{2\,(-1)^k}{(2k+3)(2k+5)}\Big(\frac{(-1)^k}{2k+3} + 2T_{k+1} - 2T_{k+1} + \frac{(-1)^{k+1}}{2k+5}\Big)\\
&= \frac{2}{2k+5}\frac{(-1)^k}{2k+3}\Big(\frac{(-1)^k}{2k+3} + 2T_{k+1}\Big) + \frac{2}{2k+3}\frac{(-1)^{k+1}}{2k+5}\Big(2T_{k+1} + \frac{(-1)^k}{2k+5}\Big)\\
&= \frac{2}{2k+5}(T_k - T_{k+1})(T_k + T_{k+1}) + \frac{2}{2k+3}(T_{k+1} - T_{k+2})(T_{k+1} + T_{k+2})\\
&= \frac{2}{2k+5}(T_k^2 - T_{k+1}^2) + \frac{2}{2k+3}(T_{k+1}^2 - T_{k+2}^2)\\
&= b_k.
\end{aligned}$$

Let

$$A_n = \sum_{k=0}^{n}(-1)^k a_k \quad \text{and} \quad B_n = \sum_{k=0}^{n}(-1)^k b_k. \tag{4}$$

Then,

$$\begin{aligned}
A_n &= \sum_{k=0}^{n}(-1)^k\Big(\frac{1}{(2k+3)^2} + \frac{1}{(2k+5)^2}\Big) + \sum_{k=0}^{n}(-1)^k\Big(\frac{1}{2k+5} - \frac{1}{2k+3}\Big)\\
&= \frac{1}{9} + \frac{(-1)^n}{(2n+5)^2} + \frac{1}{3} + \frac{(-1)^n}{2n+5} - 2\sum_{k=0}^{n}\frac{(-1)^k}{2k+3}
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
B_n &= 2\sum_{k=0}^{n}\frac{(-1)^k}{2k+5}T_k^2 + 4\sum_{k=0}^{n}\frac{(-1)^k}{(2k+3)(2k+5)}T_{k+1}^2 - 2\sum_{k=0}^{n}\frac{(-1)^k}{2k+3}T_{k+2}^2\\
&= 16\sum_{k=1}^{n}(-1)^{k+1}\frac{4k^2+8k+1}{(2k-1)(2k+1)(2k+3)(2k+5)}T_k^2\\
&\quad + 2(-1)^n\frac{4n^2+20n+17}{(2n+1)(2n+3)(2n+5)}T_{n+1}^2 + 2(-1)^{n+1}\frac{1}{2n+3}T_{n+2}^2 - \frac{22}{45}.
\end{aligned} \tag{6}$$

Since

$$\sum_{k=0}^{n}\frac{(-1)^k}{2k+3} = 1 - T_{n+1},$$

we conclude from (1) and (5) that

$$\lim_{n\to\infty} A_n = \frac{1}{9} + \frac{1}{3} - 2\Big(1 - \frac{\pi}{4}\Big) = \frac{\pi}{2} - \frac{14}{9}. \tag{7}$$

From (1) and (6) we obtain

$$\lim_{n\to\infty} B_n = 16\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4k^2+8k+1}{(2k-1)(2k+1)(2k+3)(2k+5)}T_k^2 - \frac{22}{45}. \tag{8}$$

Using (3), (4), (7) and (8) gives

$$\frac{\pi}{2} - \frac{14}{9} = -\frac{22}{45} + 16\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4k^2+8k+1}{(2k-1)(2k+1)(2k+3)(2k+5)}T_k^2.$$

Thus,

$$\pi = 2\Big(\frac{14}{9} - \frac{22}{45} - \frac{16}{15}\Big) + 32\sum_{k=0}^{\infty}(-1)^{k+1}\frac{4k^2+8k+1}{(2k-1)(2k+1)(2k+3)(2k+5)}T_k^2.$$

This leads to (2). □

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