EMMY NOETHER*

BY HERMANN WEYL

WITH deep dismay Emmy Noether's friends living in America learned about her sudden passing away on Sunday, April 14. She seemed to have got well over an operation for tumor; we thought her to be on the way to convalescence when an unexpected complication led her suddenly on the downward path to her death within a few hours. She was such a paragon of vitality, she stood on the earth so firm and healthy with a certain sturdy humor and courage for life, that nobody was prepared for this eventuality. She was at the summit of her mathematical creative power; her far-reaching imagination and her technical abilities accumulated by continued experience, had come to a perfect balance; she had eagerly set to work on new problems. And now suddenly—the end, her voice silenced, her work abruptly broken off.

"Down, down, down into the darkness of the grave
Gently they go, the beautiful, the tender, the kind;
Quietly they go, the intelligent, the witty, the brave.
I know. But I do not approve. And I am not resigned."

A mood of defiance similar to that expressed in this "Dirge without music" by Edna St. Vincent Millay, mingles with our mourning in the present hour when we are gathered to commemorate our friend, her life and work and personality.

I am not able to tell much about the outward story of her life; far from her home and those places where she lived and worked in the continuity of decades, the necessary information could not be secured. She was born the 23d of March, 1882, in the small South German university town of Erlangen. Her father was Max Noether, himself a great mathematician who played an important rôle in the development of the theory of algebraic functions as the chief representative of the algebraic-geometric school. He had come to the University of Erlangen as a professor of mathematics in 1875, and stayed there until his death in 1921. Besides Emmy there grew up in the house her brother Fritz, younger by two and a half years. He turned to applied mathematics

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in later years, was until recently professor at the Technische Hochschule in Breslau, and by the same fate that ended Emmy's career in Göttingen is now driven off to the Research Institute for Mathematics and Mechanics in Tomsk, Siberia. The Noether family is a striking example of the hereditary nature of the mathematical talent, the most shining illustration of which is the Basle Huguenot dynasty of the Bernoullis.

Side by side with Noether acted in Erlangen as a mathematician the closely befriended Gordan, an offspring of Clebsch's school like Noether himself. Gordan had come to Erlangen shortly before, in 1874, and he, too, remained associated with that university until his death in 1912. Emmy wrote her doctor's thesis under him in 1907: "On complete systems of invariants for ternary biquadratic forms"; it is entirely in line with the Gordan spirit and his problems. The Mathematische Annalen contains a detailed obituary of Gordan and an analysis of his work, written by Max Noether with Emmy's collaboration. Besides her father, Gordan must have been well-nigh one of the most familiar figures in Emmy's early life, first as a friend of the house, later as a mathematician also; she kept a profound reverence for him though her own mathematical taste soon developed in quite a different direction. I remember that his picture decorated the wall of her study in Göttingen. These two men, the father and Gordan, determined the atmosphere in which she grew up. Therefore I shall venture to describe them with a few strokes.

Riemann had developed the theory of algebraic functions of one variable and their integrals, the so-called Abelian integrals, by a function-theoretic transcendental method resting on the minimum principle of potential theory which he named after Dirichlet, and had uncovered the purely topological foundations of the manifold function-theoretic relations governing this domain. (Stringent proof of Dirichlet's principle which seemed so evident from the physicist's standpoint was only given about fifty years later by Hilbert.) There remained the task of replacing and securing his transcendental existential proofs by the explicit algebraic construction starting with the equation of the algebraic curve. Weierstrass solved this problem (in his lectures published in detail only later) in his own half function-theoretic, half algebraic way, but Clebsch had introduced Riemann's ideas into the geometric theory of algebraic curves and Noether became, after Clebsch had passed away young, his executor in this matter: he succeeded in erecting the whole structure of the algebraic geometry of curves on the basis of the so-called Noether residual theorem. This line of research was taken up later on, mainly in Italy; the vein Noether struck is still a profusely gushing spring of investigations; among us, men like Lefschetz and Zariski bear witness thereto. Later on there arose, beside Riemann's transcendental and Noether's algebraic-geometric method, an arithmetical theory of algebraic functions due to Dedekind and Weber on the one side, to Hensel and Landsberg on the other. Emmy Noether stood closer to this trend of thought. A brief report on the arithmetical theory of algebraic functions that parallels the corresponding notions in the competing theories was published by her in 1920 in the Jahresberichte der Deutschen Mathematikervereinigung. She thus supplemented the well-known report by Brill and her father on the algebraic-geometric theory that had appeared in 1894 in one of the first volumes of the Jahresberichte. Noether's residual theorem was later fitted by Emmy into her general theory of ideals in arbitrary rings. This scientific kinship of father and daughter—who became in a certain sense his successor in algebra, but stands beside him independent in her fundamental attitude and in her problems—is something extremely beautiful and gratifying. The father was—such is the impression I gather from his papers and even more from the many obituary biographies he wrote for the Mathematische Annalen—a very intelligent, warm-hearted harmonious man of many-sided interests and sterling education.

Gordan was of a different stamp. A queer fellow, impulsive and one-sided. A great walker and talker—he liked that kind of walk to which frequent stops at a beer-garden or a café belong. Either with friends, and then accompanying his discussions with violent gesticulations, completely irrespective of his surroundings; or alone, and then murmuring to himself and pondering over mathematical problems; or if in an idler mood, carrying out long numerical calculations by heart. There always remained something of the eternal "Bursche" of the 1848 type about him—an air of dressing gown, beer and tobacco, relieved however by a keen sense of humor and a strong dash of wit. When he had to listen to others, in classrooms or at meetings, he was always half asleep. As a mathematician not of Noether's rank, and of an essentially different kind. Noether himself concludes his characterization of him with the short sentence: "Er war ein Algorithmiker." His strength rested on the invention and calculative execution of formal processes. There exist papers of his where twenty pages of formulas are not interrupted by a single text word; it is told that in all his papers he himself wrote the formulas only, the text being added by his friends. Noether says of him: "The formula always and everywhere was the indispensable support for the formation of his thoughts, his conclusions and his mode of expression. . . . In his lectures he carefully avoided any fundamental definition of conceptual kind, even that of the limit."

He, too, had belonged to Clebsch's most intimate collaborators, had written with Clebsch their book on Abelian integrals; he later shifted over to the theory of invariants following his formal talent; here he added considerably to the development of the so-called symbolic method, and he finally succeeded in proving by means of this computative method of explicit construction the finiteness of a rational integral basis for binary invariants. Years later Hilbert demonstrated the theorem much more generally for an arbitrary number of variables—by an entirely new approach, the characteristic Hilbertian species of methods, putting aside the whole apparatus of symbolic treatment and attacking the thing itself as directly as possible. *Ex ungue leonem*—the young lion Hilbert showed his claws. It was, however, at first only an existential proof providing for no actual finite algebraic construction. Hence Gordan's characteristic exclamation: "This is not mathematics, but theology!" What then would he have said about his former pupil Emmy Noether's later "theology", that abhorred all calculation and operated in a much thinner air of abstraction than Hilbert ever dared!

Gordan once struck upon a formal analogy between binary invariants and the scheme of valence bonds in chemistry—the same analogy by which Sylvester had been surprised many years before when thinking about an illustration of invariant theory appropriate for an audience of laymen; it is the subject of Sylvester's paper in the first volume of the American Journal of Mathematics founded by him at Johns Hopkins. Gordan seems to have been unaware of his predecessor. Anyway, he was led by his little discovery to propose the establishment of chairs for a new science, "mathematical chemistry", all over the German universities; I mention this as an incident showing his impetuosity and lack of survey. By the way, modern quantum mechanics recently has changed this analogy into a true theory disclosing the binary invariants as the mathematical tool for describing the several valence states of a molecule in spin space.

The meteor Felix Klein, whose mathematical genius caught fire through the collision of Riemann's and Galois' worlds of ideas, skimmed Erlangen before Emmy was born; he promulgated there his "Erlanger Programm", but soon moved on to Munich. By him Gordan was inspired to those invariant theoretical investigations that center around Klein's book on the icosahedron and the adjoint questions in the theory of algebraic equations. Even after their local separation both continued in their intense cooperation—a queer contrasting team if one comes to think of Gordan's formal type and Klein's, entirely oriented by intuition. The general problem at the bottom of their endeavors, Klein's form problem has likewise stayed alive to our days and quite recently has undergone a new deep-reaching treatment by Dr. Brauer's applying to it the methods of hypercomplex number systems and their representations which formed the main field of Emmy Noether's activities during the last six or seven years.

It is queer enough that a formalist like Gordan was the mathematician from whom her mathematical orbit set out; a greater contrast is hardly imaginable than between her first paper, the dissertation, and her works of maturity; for the former is an extreme example of formal computations and the latter constitute an extreme and grandiose example of conceptual axiomatic thinking in mathematics. Her thesis ends with a table of the complete system of covariant forms for a given ternary quartic consisting of not less than 331 forms in symbolic representation. It is an awe-inspiring piece of work; but today I am afraid we should be inclined to rank it among those achievements with regard to which Gordan himself once said when asked about the use of the theory of invariants: "Oh, it is very useful indeed; one can write many theses about it."

It is not quite easy to evoke before an American audience a true picture of that state of German life in which Emmy Noether grew up in Erlangen; maybe the present generation in Germany is still more remote from it. The great stability of burgher life was in her case accentuated by the fact that Noether (and Gordan too) were settled at one university for so long an uninterrupted period. One may dare to add that the time of the primary proper impulses of their production was gone, though they undoubtedly continued to be productive mathematicians; in this regard, too, the atmosphere around her was certainly tinged by a quiet uniformity. Moreover, there belongs to the picture the high standing, and the great solidity in the recognition of, spiritual values; based on a solid education, a deep and genuine active interest in the higher achievements of intellectual culture, and on a well-developed faculty of enjoying them. There must have prevailed in the Noether home a particularly warm and companionable family life. Emmy Noether herself was, if I may say so, warm like a loaf of bread. There irradiated from her a broad, comforting, vital warmth. Our generation accuses that time of lacking all moral sincerity, of hiding behind its comfort and bourgeois peacefulness, and of ignoring the profound creative and terrible forces that really shape man's destiny; moreover of shutting its eyes to the contrast between the spirit of true Christianity which was confessed, and the private and public life as it was actually lived. Nietzsche arose in Germany as a great awakener. It is hardly possible to exaggerate the significance which Nietzsche (whom by the way Noether once met in the Engadin) had in Germany for the thorough change in the moral and mental atmosphere. I think he was fundamentally right—and yet one should not deny that in wide circles in Germany, as with the Noethers, the esteem in which the spiritual goods were held, the intellectual culture, good-heartedness, and human warmth were thoroughly genuine—notwithstanding their sentimentality, their Wagnerianism, and their plush sofas.

Emmy Noether took part in the housework as a young girl, dusted and cooked, and went to dances, and it seems her life would have been that of an ordinary woman had it not happened that just about that time it became possible in Germany for a girl to enter on a scientific career without meeting any too marked resistance. There was nothing rebellious in her nature; she was willing to accept conditions as they were. But now she became a mathematician. Her dependence on Gordan did not last long; he was important as a starting point, but was not of lasting scientific influence upon her. Nevertheless the Erlangen mathematical air may have been responsible for making her into an algebraist. Gordan retired in 1910; he was followed first by Erhard Schmidt, and the next year by Ernst Fischer. Fischer's field was algebra again, in particular the theory of elimination and of invariants. He exerted upon Emmy Noether, I believe, a more penetrating influence than Gordan did. Under his direction the transition from Gordan's formal standpoint to the Hilbert method of approach was accomplished. She refers in her papers at this time again and again to conversations with Fischer. This epoch extends until about 1919. The main interest is concentrated on finite rational and integral bases; the proof of finiteness is given by her for the invariants of a finite group (without using Hilbert's general basis theorem for ideals), for invariants with restriction to integral coefficients, and finally she attacks the same question along with the question of a minimum basis consisting of independent elements for fields of rational functions.

Already in Erlangen about 1913 Emmy lectured occasionally, substituting for her father when he was taken ill. She must have been to Göttingen about that time, too, but I suppose only on a visit with her brother Fritz. At least I remember him much better than her from my time as a Göttinger Privatdozent, 1910–1913. During the war, in 1916, Emmy came to Göttingen for good; it was due to Hilbert's and Klein's direct influence that she stayed. Hilbert at that time was over head and ears in the general theory of relativity, and for Klein, too, the theory of relativity and its connection with his old ideas of the Erlangen program brought the last flareup of his mathematical interests and mathematical production. The second volume of his history of mathematics in the nineteenth century bears witness thereof. To both Hilbert and Klein Emmy was welcome as she was able to help them with her invariant theoretic knowledge. For two of the most significant sides of the general relativity theory she gave at that time the genuine and universal mathematical formulation: First, the reduction of the problem of differential invariants to a purely algebraic one by use of "normal coordinates"; second, the identities between the left sides of Euler's equations of a problem of variation which occur when the (multiple) integral is invariant with respect to a group of transformations involving arbitrary functions (identities that contain the conservation theorem of energy and momentum in the case of invariance with respect to arbitrary transformations of the four world coordinates).

Still during the war, Hilbert tried to push through Emmy Noether's "Habilitation" in the Philosophical Faculty in Göttingen. He failed due to the resistance of the philologists and historians. It is a well-known anecdote that Hilbert supported her application by declaring at the faculty meeting, "I do not see that the sex of the candidate is an argument against her admission as Privatdozent. After all, we are a university and not a bathing establishment." Probably he provoked the adversaries even more by that remark. Nevertheless, she was able to give lectures in Göttingen, that were announced under Hilbert's name. But in 1919, after the end of the War and the proclamation of the German Republic had changed the conditions, her Habilitation became possible. In 1922 there followed her nomination as a "nichtbeamteter ausserordentlicher Professor"; this was a mere title carrying no obligations and no salary. She was, however, entrusted with a "Lehrauftrag" for algebra, which carried a modest remuneration.

During the wild times after the Revolution of 1918, she did not keep aloof from the political excitement, she sided more or less with the Social Democrats; without being actually in party life she participated intensely in the discussion of the political and social problems of the day. One of her first pupils, Grete Hermann, belonged to Nelson's philosophic-political circle in Göttingen. It is hardly imaginable nowadays how willing the young generation in Germany was at that time for a fresh start, to try to build up Germany, Europe, society in general, on the foundations of reason, humaneness, and justice. But alas! the mood among the academic youth soon enough veered around; in the struggles that shook Germany during the following years and which took on the form of civil war here and there, we find them mostly on the side of the reactionary and nationalistic forces. Responsible for this above all was the breaking by the Allies of the promise of Wilson's Fourteen Points, and the fact that Republican Germany came to feel the victors' fist not less hard than the Imperial Reich could have; in particular, the youth were embittered by the national defamation added to the enforcement of a grim peace treaty. It was then that the great opportunity for the pacification of Europe was lost, and the seed sown for the disastrous development we are the witnesses of. In later years Emmy Noether took no part in matters political. She always remained, however, a convinced pacifist, a stand which she held very important and serious.

In the modest position of a "nicht-beamteter ausserordentlicher Professor" she worked in Göttingen until 1933, during the last years in the beautiful new Mathematical Institute that had risen in Göttingen chiefly by Courant's energy and the generous financial help of the Rockefeller Foundation. I have a vivid recollection of her when I was in Göttingen as visiting professor in the winter semester of 1926–1927, and lectured on representations of continuous groups. She was in the audience; for just at that time the hypercomplex number systems and their representations had caught her interest and I remember many discussions when I walked home after the lectures, with her and von Neumann, who was in Göttingen as a Rockefeller Fellow, through the cold, dirty, rain-wet streets of Göttingen. When I was called permanently to Göttingen in 1930, I earnestly tried to obtain from the Ministerium a better position for her, because I was ashamed to occupy such a preferred position beside her whom I knew to be my superior as a mathematician in many respects. I did not succeed, nor did an attempt to push through her election as a member of the Göttinger Gesellschaft der Wissenschaften. Tradition, prejudice, external considerations, weighted the balance against her scientific merits and scientific greatness, by that time denied by no one. In my Göttingen years, 1930–1933, she was without doubt the strongest center of mathematical activity there, considering both the fertility of her scientific research program and her influence upon a large circle of pupils.

Her development into that great independent master whom we admire today was relatively slow. Such a late maturing is a rare phenomenon in mathematics; in most cases the great creative impulses lie in early youth. Sophus Lie, like Emmy Noether, is one of the few great exceptions. Not until 1920, thirteen years after her promotion, appeared in the Mathematische Zeitschrift that paper of hers written with Schmeidler, "Über Moduln in nicht-kommutativen Bereichen, insbesondere aus Differential- und Differenzen-Ausdrücken", which seems to mark the decisive turning point. It is here for the first time that the Emmy Noether appears whom we all know, and who changed the face of algebra by her work. Above all, her conceptual axiomatic way of thinking in algebra becomes first noticeable in this paper dealing with differential operators as they are quite common nowadays in quantum mechanics. In performing them, one after the other, their composition, which may be interpreted as a kind of multiplication, is not commutative. But instead of operating with the formal expressions, the simple properties of the operations of addition and multiplication to which they lend themselves are formulated as axioms at the beginning of the investigation, and these axioms then form the basis of all further reasoning. A similar procedure has remained typical for Emmy Noether from then on. Later I shall try to characterize this world of algebra as a whole in which the scene of her mathematical activities was laid.

Not less characteristic for Emmy was her collaboration with another, in this case with Schmeidler. I suppose that Schmeidler gave as much as he received in this cooperation. In later years, however, Emmy Noether frequently acted as the true originator; she was most generous in sharing her ideas with others. She had many pupils, and one of the chief methods of her research was to expound her ideas in a still unfinished state in lectures, and then discuss them with her pupils. Sometimes she lectured on the same subject one semester after another, the whole subject taking on a better ordered and more unified shape every time, and gaining of course in the substance of results. It is obvious that this method sometimes put enormous demands upon her audience. In general, her lecturing was certainly not good in technical respects. For that she was too erratic and she cared too little for a nice and well arranged form. And yet she was an inspired teacher; he who was capable of adjusting himself entirely to her, could learn very much from her. Her significance for algebra cannot be read entirely from her own papers; she had great stimulating power and many of her suggestions took final shape only in the works of her pupils or co-workers. A large part of what is contained in the second volume of van der Waerden's "Modern Algebra" must be considered her property. The same is true of parts of Deuring's recently published book on algebras in which she collaborated intensively. Hasse acknowledges that he owed the suggestion for his beautiful papers on the connection between hypercomplex quantities and the theory of class fields to casual remarks by Emmy Noether. She could just utter a far-seeing remark like this, "Norm rest symbol is nothing else than cyclic algebra" in her prophetic lapidary manner, out of her mighty imagination that hit the mark most of the time and gained in strength in the course of years; and such a remark could then become a signpost to point the way for difficult future work. And one cannot read the scope of her accomplishments from the individual results of her papers alone: she originated above all a new and epoch-making style of thinking in algebra.

She lived in close communion with her pupils; she loved them, and took interest in their personal affairs. They formed a somewhat noisy and stormy family, "the Noether boys" as we called them in Göttingen. Among her pupils proper I may name Grete Hermann, Krull, Hölzer, Grell, Koethe, Deuring, Fitting, Witt, Tsen, Shoda, Levitzki. F. K. Schmidt is strongly influenced by her, chiefly through Krull's mediation. V. d. Waerden came to her from Holland as a or more less finished mathematician and with ideas of his own; but he learned from Emmy Noether the apparatus of notions and the kind of thinking that permitted him to formulate his ideas and to solve his problems. Artin and Hasse stand beside her as two independent minds whose field of production touches on hers closely, though both have a stronger arithmetical texture. With Hasse above all she collaborated very closely during her last years. From different sides, Richard Brauer and she dealt with the profounder structural problems of algebras, she in a more abstract spirit, Brauer, educated in the school of the great algebraist I. Schur, more concretely operating with matrices and representations of groups; this, too, led to an extremely fertile cooperation. She held a rather close friendship with Alexandroff in Moscow, who came frequently as a guest to Göttingen. I believe that her mode of thinking has not been without influence upon Alexandroff's topological investigations. About 1930 she spent a semester in Moscow and there got into close touch with Pontrjagin also. Before that, in 1928–1929, she had lectured for one semester in Frankfurt while Siegel delivered a course of lectures as a visitor in Göttingen.

In the spring of 1933 the storm of the National Revolution broke over Germany. The Göttinger Mathematisch-Naturwissenschaftliche Fakultät, for the building up and consolidation of which Klein and Hilbert had worked for decades, was struck at its roots. After an interregnum of one day by Neugebauer, I had to take over the direction of the Mathematical Institute. But Emmy Noether, as well as many others, was prohibited from participation in all academic activities, and finally her venia legendi, as well as her "Lehrauftrag" and the salary going with it, were withdrawn. A stormy time of struggle like this one we spent in Göttingen in the summer of 1933 draws people closer together; thus I have a particularly vivid recollection of these months. Emmy Noether, her courage, her frankness, her unconcern about her own fate, her conciliatory spirit, were, in the midst of all the hatred and meanness, despair and sorrow surrounding us, a moral solace. It was attempted, of course, to influence the Ministerium and other responsible and irresponsible but powerful bodies so that her position might be saved. I suppose there could hardly have been in any other case such a pile of enthusiastic testimonials filed with the Ministerium as was sent in on her behalf. At that time we really fought; there was still hope left that the worst could be warded off. It was in vain. Franck, Born, Courant, Landau, Emmy Noether, Neugebauer, Bernays and others—scholars the university had before been proud of—had to go because the possibility of working was taken away from them. Göttingen scattered into the four winds! This fate brought Emmy Noether to Bryn Mawr, and the short time she taught here and as guest at our Institute for Advanced Study in Princeton is still too fresh in our memory to need to be spoken of. She harbored no grudge against Göttingen and her fatherland for what they had done to her. She broke no friendship on account of political dissension. Even last summer she returned to Göttingen, and lived and worked there as though all things were as before. She was sincerely glad that Hasse was endeavoring with success to rebuild the old, honorable and proud mathematical tradition of Göttingen even in the changed political circumstances. But she had adjusted herself with perfect ease to her new American surroundings, and her girl students here were as near to her heart as the Noether boys had been in Göttingen. She was happy at Bryn Mawr; and indeed perhaps never before in her life had she received so many signs of respect, sympathy, friendship, as were bestowed upon her during her last one and a half years at Bryn Mawr. Now we stand at her grave.

It shall not be forgotten what America did during these last two stressful years for Emmy Noether and for German science in general.

If this sketch of her life is to be followed by a short synopsis of her work and her human and scientific personality, I must attempt to draw in a few strokes the scene of her work: the world of algebra. The system of real numbers, of so paramount import for the whole of mathematics and physics, resembles a Janus head with two faces: In one aspect it is the field of the algebraic operations $+$ and $\times$, and their inversions. In the other aspect it is a continuous manifold, the parts of which are continuously connected with each other. The one is the algebraic, the other the topological face of numbers. Modern axiomatics, single-minded as it is and hence disliking this strange mixture of war and peace (in this respect differing from modern politics), carefully disjointed both parts.

Hence the pure algebraist can do nothing with his numbers except perform upon them the four species, addition, subtraction, multiplication, and division. For him, therefore, a set of numbers is closed, he has no means to get beyond it when these operations applied to any two numbers of the set always lead to a number of the same set again. Such a set is called a domain of rationality or a field. The simplest field is the set of all rational numbers. Another example is the set of the numbers of the form $a + b\sqrt{2}$ where a and b are rational, the so-called algebraic number field $(\sqrt{2})$. The classical problem of algebra is the solution of an algebraic equation $f(x) = 0$ whose coefficients may lie in a field K, for instance the field of rational numbers. Knowing a root δ of the equation, one knows at the same time all numbers arising from δ (and the numbers of K) by means of the four species: they form the algebraic field $K(\delta)$ comprising K. Within this number field $K(\delta)$, δ itself plays the rôle of a determining number from which all other numbers can be rationally derived. But many, almost all, numbers of $K(\delta)$ can take the place of δ in this respect. It is, therefore, a great advance to replace the study of the equation $f(x) = 0$ by the study of the field $K(\delta)$. We thereby extinguish unessential features, we take uniformly into account all equations arising from the one $f(x) = 0$ by rational transformations of the unknown x, and we replace a formula, the equation $f(x) = 0$, which might seduce us to blind computations, by a notion, the notion of the field which one can get at only in a conceptual way.

Within the system of *integral* numbers the operations of addition, subtraction, and multiplication only allow unlimited performance; division has to be canceled. Such a domain is called a domain of integrity or a *ring*. As the notion of integer is characteristic of number theory, one may say: number theory deals with rings instead of fields. The polynomials of one variable or indeterminate x are likewise such a domain of quantities as we described to form a ring; the coefficients of the polynomials might here be restricted to a given number field or ring. Algebra does not interpret the argument x to be a variable varying over a continuous range of values; it looks upon it as an indeterminate, an empty symbol serving only to weld the coefficients of the polynomial into a unified expression which suggests in a natural way the rules of addition and multiplication. The statement that a polynomial vanishes means that all its coefficients are zero rather than that the function takes on the value zero for all values of the independent variable. One is not forbidden to substitute an indeterminate x by a number or by a polynomial of one or several other indeterminates $y, z, \ldots$; however, this is a formal process projecting the ring of polynomials of x faithfully upon the ring of numbers or of polynomials in $y, z, \ldots$. Faithfully, that means preserving all rational relations expressible in terms of the fundamental operations, addition, subtraction, multiplication.

Besides adjunction of indeterminates, algebra knows another procedure for forming new fields or rings. Let p be a prime number, for instance 5. We take the ordinary integers, agreeing, however, to consider numbers to be equal when they are congruent mod. p, i. e., when they give the same remainder under division by p. One may illustrate this by winding the line of numbers on a circle of circumference p. A peculiar field then arises consisting of p different elements only. To the *prime number* there corresponds within the ring of polynomials of a single variable x (with numerical coefficients taken from a given number field K) the *prime polynomial* $p(x)$. By considering two polynomials equal which are congruent modulo a given prime polynomial $p(x)$, the ring of all polynomials is changed into a *field* which possesses exactly the same algebraic properties as the number field $K(\delta)$ arising from the underlying number field K by adjoining a root δ of the equation $p(x) = 0$. But the present process goes on within pure algebra without requiring solution of an equation $p(x) = 0$ that is actually unsolvable in K. This interpretation of the algebraic number fields $K(\delta)$ was given by Kronecker after Cauchy had already founded the calculation with the imaginary number i on this idea.

In such a way one was led by degrees to erect algebra in a purely axiomatic manner. A whole array of great mathematical names could be mentioned who initiated and developed this axiomatic trend: after Kronecker and Dedekind, E. H. Moore in America, Peano in Italy, Steinitz, and, above all, Hilbert in Germany. A field now is a realm of elements, called numbers, within which two operations $+$ and $\times$ are defined, satisfying the usual axioms. If one leaves out the axiom of division which states the unique invertibility of multiplication, one gets a ring instead of a field. The fields no longer appear as parts cut out of that universal realm of numbers, the continuum of the real or complex numbers that the Calculus is concerned with, but every field is now, so to speak, a world in itself. One may join the elements of any field by operations, but not the elements of different fields. This standpoint that each object which is offered to mathematical analysis carries its own kind of numbers to be defined in terms of that object and its intrinsic constituents, instead of approaching every object by the same universal number system developed à priori and independently of the applications—this standpoint, I say, has gained ground more and more also in the axiomatic foundations of geometry and recently in a rather surprising manner in quantum physics. We are here confronted by one of those mysterious parallelisms in the development of mathematics and physics that might induce one to believe in a preestablished harmony between nature and mind.

When speaking of axiomatics, I was referring to the following methodical procedure: One separates in a natural way the different sides of a concretely given object of mathematical investigation, makes each of them accessible from its own relatively narrow and easily surveyable group of assumptions, and then by joining the partial results after appropriate specialization, returns to the complex whole. The last synthetic part is purely mechanical. The art lies in the first analytical part of breaking up the whole and generalizing the parts. One does not seek the general for the sake of generality, but the point is that each generalization simplifies by reducing the hypotheses and thus makes us understand certain sides of an unsurveyable whole. Whether a partition with corresponding generalization is natural, can hardly be judged by any other criterion than its fertility. If one systematizes this procedure which the individual investigator manages supported by all the analogies available to him by the mass of his mathematical experiences and with more or less inventive ability and sensitivity, one comes upon axiomatics. Hence axiomatics is today by no means merely a method for logical clarification and deepening of the foundations, but it has become a powerful weapon of concrete mathematical research itself. This method was applied by Emmy Noether with masterly skill, it suited her nature, and she made algebra the Eldorado of axiomatics. An important point is the ascertainment of the "right" general notions like field, ring, ideal, etc., the splitting-up of a proposition into partial propositions and their right generalizations by means of those general notions. This partition of the whole and screening off of the unessential features once accomplished, the proof of the individual steps does not cause any serious trouble in many cases. In a conference on topology and abstract algebra as two ways of mathematical understanding, in 1931, I said this:

"Nevertheless I should not pass over in silence the fact that today the feeling among mathematicians is beginning to spread that the fertility of these abstracting methods is approaching exhaustion. The case is this: that all these nice general notions do not fall into our laps by themselves. But definite concrete problems were first conquered in their undivided complexity, singlehanded by brute force, so to speak. Only afterwards the axiomaticians came along and stated: Instead of breaking in the door with all your might and bruising your hands, you should have constructed such and such a key of skill, and by it you would have been able to open the door quite smoothly. But they can construct the key only because they are able, after the breaking in was successful, to study the lock from within and without. Before you can generalize, formalize and axiomatize, there must be a mathematical substance. I think that the mathematical substance in the formalizing of which we have trained ourselves during the last decades, becomes gradually exhausted. And so I foresee that the generation now rising will have a hard time in mathematics."

Emmy Noether protested against that: and indeed she could point to the fact that just during the last years the axiomatic method had disclosed in her hands new, concrete, profound problems by the application of non-commutative algebra upon commutative fields and their number theory, and had shown the way to their solution.

Emmy Noether's scientific production seems to me to fall into three clearly distinct epochs: (1) the period of relative dependence, 1907–1919; (2) the investigations grouped around the general theory of ideals, 1920–26; (3) the study of the non-commutative algebras, their representations by linear transformations, and their application to the study of commutative number fields and their arithmetics, from 1927 on. The first epoch was described in the sketch of her life. I should now like to say a few words about the second epoch, the epoch of the general theory of ideals.

The ideals had been devised by Dedekind in order to reestablish, by introducing appropriate ideal elements, the main law of unique decomposition of a number into prime factors that broke down in algebraic number fields. The thought consisted in replacing a number, like 6 for instance, in its property as a divisor by the set of all numbers divisible by 6; this set is called the ideal (6). In the same manner one may interpret the greatest common divisor of two numbers, a, b, as the set of all numbers of form $ax + by$ where x, y range independently over all integers. In the ring of ordinary integers this system is identical with a system of the multiples of a single number d, the greatest common divisor. This, however, is not the case in algebraic number fields, and hence it becomes necessary to admit as divisors not only numbers but also ideals. An ideal in a ring R then has to be defined as a subset of R such that sum and difference of two numbers of the ideal belong to the ideal as well as the product of a number of the ideal by an arbitrary number of the ring. Still, from another side, this notion appeared in algebraic geometry. An algebraic surface in space is defined by one algebraic equation $f = 0$; here f is a polynomial with respect to the coordinates. If one is to consider algebraic manifolds of fewer dimensions, one has to put down instead a finite system of algebraic equations $f_1 = 0, f_2 = 0, \ldots, f_h = 0$. But then all polynomials vanish upon the algebraic manifold which arise by linear combination of the basic polynomials $f_1, f_2, \ldots, f_h$ in the form $A_1f_1 + A_2f_2 + \ldots + A_hf_h$ where the

A's are quite arbitrary polynomials. All the polynomials of this kind form an ideal in the ring of polynomials; the algebraic manifold consists of the points in which all polynomials of the ideal vanish. With such ideals Hilbert's basis theorem was concerned, one of the chief tools in Hilbert's study of invariants; it asserts that every ideal of polynomials has a finite basis. Noether's residual theorem contains a criterion that allows us to decide whether a polynomial belongs to an ideal the members of which have in common only a finite number of zeros. For ideals of polynomials Lasker—better known to non-mathematicians as world chess champion for many years—obtained results which showed that their laws depart considerably from those met by Dedekind in the algebraic number fields.

Consider, for instance, the following three rings: the ring of ordinary integers, the rings of polynomials of one and of two independent variables with rational coefficients. The theorem of unique decomposition into prime factors holds in each of them; but Euclid's algorithm or the fact that the greatest common divisor of two elements, a, b, is contained in the ideal (a, b), i. e., can be expressed in the form $af + bg$ by means of two appropriate elements, f, g, of the ring, is true only in the first two cases. Indeed, in the domain of polynomials of two indeterminates x and y, the polynomials x and y themselves have no common divisor; nevertheless an equation like $1 = xf + yg$ where f and g are two polynomials, is impossible as the right side vanishes at the origin $x = 0, y = 0$.

Emmy Noether developed a general theory of ideals on an axiomatic basis that comprised all cases. Her chief axiom is the Teilerkettensatz: the hypothesis that a chain of ideals $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \ldots$ necessarily comes to an end after a finite number of steps if each term $\mathfrak{a}_i$ comprises the preceding $\mathfrak{a}_{i-1}$ as a proper part. By her abstract theory many important developments of mathematics are welded together. Moreover, she showed how one can descend in the same axiomatic manner to the polynomial ideals on the one hand, and to the classical case of ideals in algebraic number fields on the other hand. In some instances her general theory passes even beyond what was known before through Lasker for polynomial ideals.

Until now we have stuck to all axioms satisfied by the ordinary numbers. There exist, however, strong motives for abandoning the commutative law of multiplication. Indeed, operations like the rotations of a rigid body in space, are entities which behave with respect to their composition in a non-commutative fashion: for the composition of two rotations it really matters whether one first performs the first and then the second, or does it in inverse order. Composition is here considered as a kind of multiplication. Rotations when expressed in terms of coordinates are linear transformations. The linear transformations, as they are capable of addition and composition or multiplication, form the most important example of non-commutative quantities. One therefore attempts to realize any given abstract non-commutative ring or "algebra" of quantities by linear transformations without destroying the relations established among them by the fundamental operations + and ×; this is the aim of the theory of representations. The theory of non-commutative algebras and their representations was built up by Emmy Noether in a new unified, purely conceptual manner by making use of all results that had been accumulated by the ingenious labors of decades through Molien, Frobenius, Dickson, Wedderburn, and others. The notion of the ideal in several new versions again plays the decisive part. Besides it, the idea of *automorphism* proves to be rather useful, i. e., of those mappings one can perform within an algebra without destroying the internal relations. Calculative tools are discarded like, for instance, a certain determinant the non-vanishing of which Dedekind had used as a criterion for semi-simplicity; this was the more desirable as this criterion fails in some domains of rationality. In intense cooperation with Hasse and with Brauer she investigated the structure of non-commutative algebras and applied the theory by means of her *verschränktes Produkt* (cross product) to the ordinary commutative number fields and their arithmetics. The most important papers of this epoch are "Hyperkomplexe Grössen und Darstellungstheorie", 1929; "Nicht-kommutative Algebra", 1933; and three smaller papers about norm rests and the principal genus theorem. Her theory of cross products was published by Hasse in connection with his investigations about the theory of cyclic algebras. A common paper by Brauer, Hasse, and Emmy Noether proving the fact that every simple algebra over an ordinary algebraic number field is cyclic in Dickson's sense, will remain a high mark in the history of algebra.

I must forego giving a picture of the content of these profound investigations. Instead, I had better try to close with a short general estimate of Emmy Noether as a mathematician and as a personality.

Her strength lay in her ability to operate abstractly with concepts. It was not necessary for her to allow herself to be led to new results on the leading strings of known concrete examples. This had the disadvantage, however, that she was sometimes but incompletely cognizant of the specific details of the more interesting applications of her general theories. She possessed a most vivid imagination, with the aid of which she could visualize remote connections; she constantly strove toward unification. In this she sought out the essentials in the known facts.

brought them into order by means of appropriate general concepts, espied the vantage point from which the whole could best be surveyed, cleansed the object under consideration of superfluous dross, and thereby won through to so simple and distinct a form that the venture into new territory could be undertaken with the greatest prospect of success. This clarifying power she proved, for example, in her theory of the cross product, in which almost all the facts had already been found by Dickson and by Brauer. She possessed a strong drive toward axiomatic purity. All should be accomplished within the frame and with the aid of the intrinsic properties of the structure under investigation; nothing should be brought from without, and only invariant processes should be applied. Thus it seemed to her that the use of matrices which commute with all the elements of a given matric algebra, so often to be found in the work of Schur, was inappropriate; accordingly she used the automorphisms instead. This can be carried too far, however, as when she disdained to employ a primitive element in the development of the Galois theory. She once said:

"If one proves the equality of two numbers a and b by showing first that $a \leqq b$ and then $a \geqq b$, it is unfair; one should instead show that they are really equal by disclosing the inner ground for their equality."

Of her predecessors in algebra and number theory, Dedekind was most closely related to her. For him she felt a deep veneration. She expected her students to read Dedekind's appendices to Dirichlet's "Zahlentheorie" not only in one, but in all editions. She took a most active part in the editing of Dedekind's works; here the attempt was made to indicate, after each of Dedekind's papers, the modern development built upon his investigations. Her affinity with Dedekind, who was perhaps the most typical Lower Saxon among German mathematicians, proves by a glaring example how illusory it is to associate in a schematic way race with the style of mathematical thought. In addition to Dedekind's work, that of Steinitz on the theory of abstract fields was naturally of great importance for her own work. She lived through a great flowering of algebra in Germany, toward which she contributed much. Her methods need not, however, be considered the only means of salvation. In addition to Artin and Hasse, who in some respects are akin to her, there are algebraists of a still more different stamp, such as I. Schur in Germany, Dickson and Wedderburn in America, whose achievements are certainly not behind hers in depth and significance. Perhaps her followers, in pardonable enthusiasm, have not always fully recognized this fact.

Emmy Noether was a zealous collaborator in the editing of the

Mathematische Annalen. That this work was never explicitly recognized may have caused her some pain.

It was only too easy for those who met her for the first time, or had no feeling for her creative power, to consider her queer and to make fun at her expense. She was heavy of build and loud of voice, and it was often not easy for one to get the floor in competition with her. She preached mightily, and not as the scribes. She was a rough and simple soul, but her heart was in the right place. Her frankness was never offensive in the least degree. In everyday life she was most unassuming and utterly unselfish; she had a kind and friendly nature. Nevertheless she enjoyed the recognition paid her; she could answer with a bashful smile like a young girl to whom one had whispered a compliment. No one could contend that the Graces had stood by her cradle; but if we in Göttingen often chaffingly referred to her as "der Noether" (with the masculine article), it was also done with a respectful recognition of her power as a creative thinker who seemed to have broken through the barrier of sex. She possessed a rare humor and a sense of sociability; a tea in her apartments could be most pleasurable. But she was a one-sided being who was thrown out of balance by the overweight of her mathematical talent. Essential aspects of human life remained undeveloped in her, among them, I suppose, the erotic, which, if we are to believe the poets, is for many of us the strongest source of emotions, raptures, desires, and sorrows, and conflicts. Thus she sometimes gave the impression of an unwieldly child, but she was a kind-hearted and courageous being, ready to help, and capable of the deepest loyalty and affection. And of all I have known, she was certainly one of the happiest.

Comparison with the other woman mathematician of world renown, Sonya Kovalevskaya, suggests itself. Sonya had certainly the more complete personality, but was also of a much less happy nature. In order to pursue her studies Sonya had to defy the opposition of her parents, and entered into a marriage in name only, although it did not quite remain so. Emmy Noether had, as I have already indicated, neither a rebellious nature nor Bohemian leanings. Sonya possessed feminine charm, instincts, and vanity; social successes were by no means immaterial to her. She was a creature of tension and whims; mathematics made her unhappy, whereas Emmy found the greatest pleasure in her work. Sonya followed literary pursuits outside of mathematics. In her later years in Paris, as she worked feverishly on a paper to be submitted for a mathematical prize, Sonya, alluding in a letter to a certain M. with whom she was in love, wrote "The fat M. occupies all the room on my couch and in my thoughts." Such was Sonya: you see the tension between her creative mind and life with its passion and the self-mocking spirit ironically viewing her own desperate conflict. How far from Emmy's possibilities! But Emmy Noether without doubt possessed by far the greater power, the greater scientific talent.

Indeed, two traits determined above all her nature: First, the native productive power of her mathematical genius. She was not clay, pressed by the artistic hands of God into a harmonious form, but rather a chunk of human primary rock into which he had blown his creative breath of life. Second, her heart knew no malice; she did not believe in evil—indeed it never entered her mind that it could play a rôle among men. This was never more forcefully apparent to me than in the last stormy summer, that of 1933, which we spent together in Göttingen. The memory of her work in science and of her personality among her fellows will not soon pass away. She was a great mathematician, the greatest, I firmly believe, that her sex has ever produced, and a great woman.

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Das ebd. S. 475 f. angeschlossene Schriftenverzeichnis wurde hier weggelassen (vgl. S. 40 ff.).

Nachruf von H. Weyl, S. 53–72, aus: Scripta mathematica III, 3 (1935), S. 201–220.